Projective Splitting as a Warped Proximal Algorithm

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Abstract. We show that the asynchronous block-iterative primal-dual projective splitting framework introduced by P. L. Combettes and J. Eckstein in their 2018 *Math. Program.* paper can be viewed as an instantiation of the recently proposed warped proximal algorithm.

Keywords. Warped proximal algorithm, projective splitting, primal-dual algorithm, splitting algorithm, monotone inclusion, monotone operator.

In [4], the warped proximal algorithm was proposed and its pertinence was illustrated through the ability to unify existing methods such as those of [1, 6, 10, 11], and to design novel flexible algorithms for solving challenging monotone inclusions. Let us state a version of [4, Theorem 4.2].

Proposition 1 *Let* H *be a real Hilbert space, let* $\mathsf{M}\colon \mathsf{H} \to 2^{\mathsf{H}}$ *be a maximally monotone operator such that* $\operatorname{zer}\mathsf{M} \neq \varnothing$*, let* $\mathsf{x}_0 \in \mathsf{H}$*, let* $\varepsilon \in \,]0,1[$*, let* $\alpha \in \,]0,+\infty[$*, and let* $\beta \in [\alpha,+\infty[$*. For every* $n \in \mathbb{N}$*, let* $\mathsf{K}_n\colon \mathsf{H} \to \mathsf{H}$ *be* α*-strongly monotone and* β*-Lipschitzian, and let* $\lambda_n \in [\varepsilon, 2-\varepsilon]$*. Iterate*

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{take } \widetilde{\mathsf{x}}_n \in \mathsf{H}\\
\mathsf{y}_n = (\mathsf{K}_n + \mathsf{M})^{-1}(\mathsf{K}_n\widetilde{\mathsf{x}}_n)\\
\mathsf{y}_n^* = \mathsf{K}_n\widetilde{\mathsf{x}}_n - \mathsf{K}_n\mathsf{y}_n\\
\text{if } \langle \mathsf{x}_n - \mathsf{y}_n \mid \mathsf{y}_n^*\rangle > 0\\
\left\lfloor \mathsf{x}_{n+1} = \mathsf{x}_n - \dfrac{\lambda_n\langle \mathsf{x}_n - \mathsf{y}_n \mid \mathsf{y}_n^*\rangle}{\|\mathsf{y}_n^*\|^2}\,\mathsf{y}_n^* \right.\\
\text{else}\\
\left\lfloor \mathsf{x}_{n+1} = \mathsf{x}_n. \right.
\end{array}
\right.
\end{array}
\tag{1}
$$

Then the following hold:

(i) $(\mathsf{x}_n)_{n\in\mathbb{N}}$ *is bounded.*

(ii) $\sum_{n\in\mathbb{N}} \|\mathsf{x}_{n+1} - \mathsf{x}_n\|^2 < +\infty$.

(iii) $(\forall n \in \mathbb{N})\ \langle \mathsf{x}_n - \mathsf{y}_n \mid \mathsf{y}_n^*\rangle \leqslant \varepsilon^{-1}\|\mathsf{y}_n^*\|\,\|\mathsf{x}_{n+1} - \mathsf{x}_n\|$.

(iv) *Suppose that* $\widetilde{\mathsf{x}}_n - \mathsf{x}_n \to \mathbf{0}$*. Then* $(\mathsf{x}_n)_{n\in\mathbb{N}}$ *converges weakly to a point in* $\operatorname{zer}\mathsf{M}$.

Proof. We deduce from [4, Proposition 3.9(i)[d]&(ii)[b]] that (1) is a special case of [4, Eq. (4.5)].

(i): An inspection of the proof of [4, Theorem 4.2] reveals that $(\mathsf{x}_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to $\operatorname{zer}\mathsf{M}$, that is, $(\forall \mathsf{z} \in \operatorname{zer}\mathsf{M})(\forall n \in \mathbb{N})\ \|\mathsf{x}_{n+1} - \mathsf{z}\| \leqslant \|\mathsf{x}_n - \mathsf{z}\|$. Therefore, the boundedness of $(\mathsf{x}_n)_{n\in\mathbb{N}}$ follows from [3, Proposition 5.4(i)].

(ii): [4, Theorem 4.2(i)].

(iii): [4, Eqs. (4.8), (4.9), and (4.4)].

(iv): Combine [4, Theorem 4.2(ii)] and [4, Remark 4.3]. ☐

A problem of interest in modern nonlinear analysis is the following (see, e.g., [1, 5, 6, 7] and the references therein for discussions on this problem).

Problem 2 Let $(\mathcal{H}_i)_{i\in I}$ and $(\mathcal{G}_k)_{k\in K}$ be finite families of real Hilbert spaces. For every $i \in I$ and every $k \in K$, let $A_i\colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ and $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ be maximally monotone, let $z_i^* \in \mathcal{H}_i$, let $r_k \in \mathcal{G}_k$, and let $L_{k,i}\colon \mathcal{H}_i \to \mathcal{G}_k$ be linear and bounded. The problem is to

$$\text{find } (\overline{x}_i)_{i\in I} \in \bigtimes_{i\in I} \mathcal{H}_i \text{ and } (\overline{v}_k^*)_{k\in K} \in \bigtimes_{k\in K} \mathcal{G}_k \text{ such that } \begin{cases} (\forall i \in I) \;\; z_i^* - \sum_{k\in K} L_{k,i}^* \overline{v}_k^* \in A_i \overline{x}_i \\ (\forall k \in K) \;\; \sum_{i\in I} L_{k,i} \overline{x}_i - r_k \in B_k^{-1} \overline{v}_k^*. \end{cases} \tag{2}$$

The set of solutions to (2) is denoted by $\mathbf{Z}$.

The first asynchronous block-iterative algorithm to solve Problem 2 was proposed in [7, Algorithm 12] as an extension of the projective splitting techniques found in [1, 8]. The goal of this short note is to interpret these projective splitting frameworks in simple terms as warped proximal iterations. More precisely, we show that [7, Algorithm 12] can be viewed as an instantiation of (1). To this end, we first derive an abstract weak convergence principle from Proposition 1. (We refer the reader to [3] for background on monotone operator theory and nonlinear analysis.)

Theorem 3 *Let $\mathbf{H}$ be a real Hilbert space, let $\mathbf{A}\colon \mathbf{H} \to 2^{\mathbf{H}}$ be a maximally monotone operator, and let $\mathbf{S}\colon \mathbf{H} \to \mathbf{H}$ be a bounded linear operator such that $\mathbf{S}^* = -\mathbf{S}$. In addition, let $\mathbf{x}_0 \in \mathbf{H}$, let $\varepsilon \in \left]0,1\right[$, let $\alpha \in \left]0,+\infty\right[$, let $\rho \in \left[\alpha,+\infty\right[$, and for every $n \in \mathbb{N}$, let $\mathbf{F}_n\colon \mathbf{H} \to \mathbf{H}$ be α-strongly monotone and ρ-Lipschitzian, and let $\lambda_n \in [\varepsilon, 2-\varepsilon]$. Iterate*

$$\begin{array}{l} \text{for } n = 0,1,\ldots \\ \left\lfloor \begin{array}{l} \text{take } \mathbf{u}_n \in \mathbf{H},\ \mathbf{e}_n^* \in \mathbf{H},\ \text{and } \mathbf{f}_n^* \in \mathbf{H} \\ \mathbf{u}_n^* = \mathbf{F}_n\mathbf{u}_n - \mathbf{S}\mathbf{u}_n + \mathbf{e}_n^* + \mathbf{f}_n^* \\ \mathbf{y}_n = (\mathbf{F}_n + \mathbf{A})^{-1}\mathbf{u}_n^* \\ \mathbf{a}_n^* = \mathbf{u}_n^* - \mathbf{F}_n\mathbf{y}_n \\ \mathbf{y}_n^* = \mathbf{a}_n^* + \mathbf{S}\mathbf{y}_n \\ \pi_n = \langle \mathbf{x}_n \mid \mathbf{y}_n^* \rangle - \langle \mathbf{y}_n \mid \mathbf{a}_n^* \rangle \\ \text{if } \pi_n > 0 \\ \left\lfloor \begin{array}{l} \tau_n = \|\mathbf{y}_n^*\|^2 \\ \theta_n = \lambda_n \pi_n / \tau_n \\ \mathbf{x}_{n+1} = \mathbf{x}_n - \theta_n \mathbf{y}_n^* \end{array} \right. \\ \text{else} \\ \left\lfloor \; \mathbf{x}_{n+1} = \mathbf{x}_n. \right. \end{array} \right. \end{array} \tag{3}$$

Suppose that $\operatorname{zer}(\mathbf{A} + \mathbf{S}) \neq \varnothing$. *Then the following hold:*

(i) $\sum_{n\in\mathbb{N}} \|\mathbf{x}_{n+1} - \mathbf{x}_n\|^2 < +\infty$.

(ii) *Suppose that $\mathbf{u}_n - \mathbf{x}_n \to \mathbf{0}$, that $\mathbf{e}_n^* \to \mathbf{0}$, that $(\mathbf{f}_n^*)_{n\in\mathbb{N}}$ is bounded, and that there exists $\delta \in \left]0,1\right[$ such that*

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{f}_n^* \rangle \geqslant -\delta \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n \rangle \\ \langle \mathbf{a}_n^* + \mathbf{S}\mathbf{u}_n - \mathbf{e}_n^* \mid \mathbf{f}_n^* \rangle \leqslant \delta \|\mathbf{a}_n^* + \mathbf{S}\mathbf{u}_n - \mathbf{e}_n^*\|^2. \end{cases} \tag{4}$$

Then $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\operatorname{zer}(\mathbf{A} + \mathbf{S})$.

Proof. Set $\mathbf{M} = \mathbf{A} + \mathbf{S}$ and $(\forall n \in \mathbb{N})$ $\mathbf{K}_n = \mathbf{F}_n - \mathbf{S}$. Then, it follows from [3, Example 20.35 and Corollary 25.5(i)] that $\mathbf{M}$ is maximally monotone with $\operatorname{zer}\mathbf{M} \neq \varnothing$. Now take $n \in \mathbb{N}$. We have

$$\mathbf{K}_n + \mathbf{M} = \mathbf{F}_n + \mathbf{A}. \tag{5}$$

Since $\mathbf{S}^* = -\mathbf{S}$, we deduce that

$$\mathbf{K}_n \text{ is } \alpha\text{-strongly monotone and } \beta\text{-Lipschitzian}, \tag{6}$$

where $\beta = \rho + \|\mathbf{S}\|$. Thus, [3, Corollary 20.28 and Proposition 22.11(ii)] guarantee that there exists $\widetilde{\mathbf{x}}_n \in \mathbf{H}$ such that

$$\mathbf{u}_n^* = \mathbf{K}_n\widetilde{\mathbf{x}}_n. \tag{7}$$

Hence, by (3) and (5),

$$\mathbf{y}_n = (\mathbf{K}_n + \mathbf{M})^{-1}(\mathbf{K}_n\widetilde{\mathbf{x}}_n) \quad \text{and} \quad \mathbf{y}_n^* = \mathbf{u}_n^* - \mathbf{F}_n\mathbf{y}_n + \mathbf{S}\mathbf{y}_n = \mathbf{K}_n\widetilde{\mathbf{x}}_n - \mathbf{K}_n\mathbf{y}_n. \tag{8}$$

At the same time, we have $\langle \mathbf{y}_n \mid \mathbf{S}\mathbf{y}_n\rangle = 0$ and it thus results from (3) that

$$\pi_n = \langle \mathbf{x}_n \mid \mathbf{y}_n^*\rangle - \langle \mathbf{y}_n \mid \mathbf{a}_n^* + \mathbf{S}\mathbf{y}_n\rangle = \langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^*\rangle. \tag{9}$$

Altogether, (3) is a special case of (1).

(i): Proposition 1(ii).

(ii): In the light of Proposition 1(iv), it suffices to verify that $\widetilde{\mathbf{x}}_n - \mathbf{x}_n \to \mathbf{0}$. For every $n \in \mathbb{N}$, since $\mathbf{K}_n + \mathbf{M}$ is maximally monotone [3, Corollary 25.5(i)] and α-strongly monotone, [3, Example 22.7 and Proposition 22.11(ii)] implies that $(\mathbf{K}_n + \mathbf{M})^{-1}\colon \mathbf{H} \to \mathbf{H}$ is $(1/\alpha)$-Lipschitzian. Therefore, we derive from (3), (5), [4, Proposition 3.10(i)], and (6) that $(\forall \mathbf{z} \in \operatorname{zer}\mathbf{M})(\forall n \in \mathbb{N})$ $\alpha\|\mathbf{y}_n - \mathbf{z}\| = \alpha\|(\mathbf{K}_n + \mathbf{M})^{-1}\mathbf{u}_n^* - (\mathbf{K}_n + \mathbf{M})^{-1}(\mathbf{K}_n\mathbf{z})\| \leqslant \|\mathbf{u}_n^* - \mathbf{K}_n\mathbf{z}\| = \|\mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{z} + \mathbf{e}_n^* + \mathbf{f}_n^*\| \leqslant \|\mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{z}\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\| \leqslant \beta\|\mathbf{u}_n - \mathbf{z}\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\|$. Thus, since Proposition 1(i) and our assumption imply that $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded, it follows that $(\mathbf{y}_n)_{n\in\mathbb{N}}$ is bounded. At the same time, for every $n \in \mathbb{N}$, we get from (3) that

$$\mathbf{y}_n^* = \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{e}_n^* + \mathbf{f}_n^* - (\mathbf{S}\mathbf{u}_n - \mathbf{S}\mathbf{y}_n) = \mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{y}_n + \mathbf{e}_n^* + \mathbf{f}_n^* \tag{10}$$

and, thus, from (6) that $\|\mathbf{y}_n^*\| \leqslant \|\mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{y}_n\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\| \leqslant \beta\|\mathbf{u}_n - \mathbf{y}_n\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\|$. Thus, $(\mathbf{y}_n^*)_{n\in\mathbb{N}}$ is bounded, from which, (i), and Proposition 1(iii) we obtain $\overline{\lim}\langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^*\rangle \leqslant 0$. In turn, since $\mathbf{x}_n - \mathbf{u}_n \to \mathbf{0}$ and $\mathbf{e}_n^* \to \mathbf{0}$, it results from (10) and (4) that

$$\begin{aligned}
0 &\geqslant \overline{\lim}\langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^*\rangle\\
&= \overline{\lim}\left(\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{y}_n^*\rangle + \langle \mathbf{x}_n - \mathbf{u}_n \mid \mathbf{y}_n^*\rangle\right)\\
&= \overline{\lim}\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{y}_n^*\rangle\\
&= \overline{\lim}\left(\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{e}_n^* + \mathbf{f}_n^*\rangle - \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{S}\mathbf{u}_n - \mathbf{S}\mathbf{y}_n\rangle\right)\\
&= \overline{\lim}\left(\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{f}_n^*\rangle + \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{e}_n^*\rangle\right)\\
&\geqslant \overline{\lim}\left((1-\delta)\langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n\rangle + \langle \mathbf{u}_n - \mathbf{y}_n \mid \mathbf{e}_n^*\rangle\right)\\
&\geqslant \overline{\lim}\,\alpha(1-\delta)\|\mathbf{u}_n - \mathbf{y}_n\|^2\\
&\geqslant \overline{\lim}\,\alpha(1-\delta)\rho^{-2}\|\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n\|^2.
\end{aligned} \tag{11}$$

Hence, $\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n \to \mathbf{0}$. On the other hand, since $(\mathbf{f}_n^*)_{n\in\mathbb{N}}$ is bounded and since (3) yields $(\mathbf{a}_n^* + \mathbf{S}\mathbf{u}_n - \mathbf{e}_n^*)_{n\in\mathbb{N}} = (\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{f}_n^*)_{n\in\mathbb{N}}$, we derive from (4) that

$$\overline{\lim}(1-\delta)\|\mathbf{f}_n^*\|^2 = \overline{\lim}\left(\langle \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n \mid \mathbf{f}_n^*\rangle + (1-\delta)\|\mathbf{f}_n^*\|^2\right)$$

$$
\begin{aligned}
&= \overline{\lim}\left(\langle \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{f}_n^* \mid \mathbf{f}_n^*\rangle - \delta\|\mathbf{f}_n^*\|^2\right)\\
&\leqslant \overline{\lim}\left(\delta\|\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n + \mathbf{f}_n^*\|^2 - \delta\|\mathbf{f}_n^*\|^2\right)\\
&= \overline{\lim}\left(\delta\|\mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n\|^2 + 2\delta\langle \mathbf{F}_n\mathbf{u}_n - \mathbf{F}_n\mathbf{y}_n \mid \mathbf{f}_n^*\rangle\right)\\
&= 0.
\end{aligned}
\tag{12}
$$

Therefore, $\mathbf{f}_n^* \to \mathbf{0}$. Consequently, by (6), (7), and (3), $\alpha\|\widetilde{\mathbf{x}}_n - \mathbf{x}_n\| \leqslant \|\mathbf{K}_n\widetilde{\mathbf{x}}_n - \mathbf{K}_n\mathbf{x}_n\| = \|\mathbf{K}_n\mathbf{u}_n - \mathbf{K}_n\mathbf{x}_n + \mathbf{e}_n^* + \mathbf{f}_n^*\| \leqslant \beta\|\mathbf{u}_n - \mathbf{x}_n\| + \|\mathbf{e}_n^*\| + \|\mathbf{f}_n^*\| \to 0$. □

We are now ready to recover [7, Theorem 13]; see also [7, Remark 4] for comments on the error sequences $(e_{i,n})_{n\in\mathbb{N},i\in I_n}$ and $(f_{k,n})_{n\in\mathbb{N},k\in K_n}$ in (15). The reader is referred to [7] for discussions on the features of the algorithm (15). Recall that, given a real Hilbert space $\mathcal{H}$ with identity operator Id, the resolvent of an operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is $J_A = (\mathrm{Id} + A)^{-1}$.

Corollary 4 ([7]) *Consider the setting of Problem 2 and suppose that $\mathbf{Z} \neq \varnothing$. Let $(I_n)_{n\in\mathbb{N}}$ be nonempty subsets of I and $(K_n)_{n\in\mathbb{N}}$ be nonempty subsets of K such that*

$$
I_0 = I, \quad K_0 = K, \quad \textit{and} \quad (\exists T \in \mathbb{N})(\forall n \in \mathbb{N}) \ \bigcup_{j=n}^{n+T} I_j = I \ \textit{and} \ \bigcup_{j=n}^{n+T} K_j = K. \tag{13}
$$

In addition, let $D \in \mathbb{N}$, let $\varepsilon \in \left]0,1\right[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be in $[\varepsilon, 2-\varepsilon]$, and for every $i \in I$ and every $k \in K$, let $(c_i(n))_{n\in\mathbb{N}}$ and $(d_k(n))_{n\in\mathbb{N}}$ be in $\mathbb{N}$ such that

$$
(\forall n \in \mathbb{N}) \quad n - D \leqslant c_i(n) \leqslant n \quad \textit{and} \quad n - D \leqslant d_k(n) \leqslant n, \tag{14}
$$

let $(\gamma_{i,n})_{n\in\mathbb{N}}$ *and* $(\mu_{k,n})_{n\in\mathbb{N}}$ *be in* $[\varepsilon,1/\varepsilon]$, *let* $x_{i,0}\in\mathcal{H}_i$, *and let* $v^*_{k,0}\in\mathcal{G}_k$. *Iterate*

$$
\begin{array}{l}
\text{for } n=0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for every } i\in I_n\\
\left\lfloor
\begin{array}{l}
\text{take } e_{i,n}\in\mathcal{H}_i\\
l^*_{i,n}=\sum_{k\in K}L^*_{k,i}v^*_{k,c_i(n)}\\
a_{i,n}=J_{\gamma_{i,c_i(n)}A_i}\big(x_{i,c_i(n)}+\gamma_{i,c_i(n)}(z^*_i-l^*_{i,n})+e_{i,n}\big)\\
a^*_{i,n}=\gamma^{-1}_{i,c_i(n)}(x_{i,c_i(n)}-a_{i,n}+e_{i,n})-l^*_{i,n}
\end{array}
\right.\\
\text{for every } i\in I\smallsetminus I_n\\
\left\lfloor
\begin{array}{l}
a_{i,n}=a_{i,n-1}\\
a^*_{i,n}=a^*_{i,n-1}
\end{array}
\right.\\
\text{for every } k\in K_n\\
\left\lfloor
\begin{array}{l}
\text{take } f_{k,n}\in\mathcal{G}_k\\
l_{k,n}=\sum_{i\in I}L_{k,i}x_{i,d_k(n)}\\
b_{k,n}=r_k+J_{\mu_{k,d_k(n)}B_k}\big(l_{k,n}+\mu_{k,d_k(n)}v^*_{k,d_k(n)}+f_{k,n}-r_k\big)\\
b^*_{k,n}=v^*_{k,d_k(n)}+\mu^{-1}_{k,d_k(n)}(l_{k,n}-b_{k,n}+f_{k,n})\\
t_{k,n}=b_{k,n}-\sum_{i\in I}L_{k,i}a_{i,n}
\end{array}
\right.\\
\text{for every } k\in K\smallsetminus K_n\\
\left\lfloor
\begin{array}{l}
b_{k,n}=b_{k,n-1}\\
b^*_{k,n}=b^*_{k,n-1}\\
t_{k,n}=b_{k,n}-\sum_{i\in I}L_{k,i}a_{i,n}
\end{array}
\right.\\
\text{for every } i\in I\\
\lfloor\; t^*_{i,n}=a^*_{i,n}+\sum_{k\in K}L^*_{k,i}b^*_{k,n}\\
\pi_n=\sum_{i\in I}\big(\langle x_{i,n}\mid t^*_{i,n}\rangle-\langle a_{i,n}\mid a^*_{i,n}\rangle\big)+\sum_{k\in K}\big(\langle t_{k,n}\mid v^*_{k,n}\rangle-\langle b_{k,n}\mid b^*_{k,n}\rangle\big)\\
\text{if } \pi_n>0\\
\left\lfloor
\begin{array}{l}
\tau_n=\sum_{i\in I}\|t^*_{i,n}\|^2+\sum_{k\in K}\|t_{k,n}\|^2\\
\theta_n=\lambda_n\pi_n/\tau_n
\end{array}
\right.\\
\text{else}\\
\lfloor\; \theta_n=0\\
\text{for every } i\in I\\
\lfloor\; x_{i,n+1}=x_{i,n}-\theta_n t^*_{i,n}\\
\text{for every } k\in K\\
\lfloor\; v^*_{k,n+1}=v^*_{k,n}-\theta_n t_{k,n}.
\end{array}
\right.
\end{array}
\tag{15}
$$

In addition, suppose that there exist $\eta\in\,]0,+\infty[$, $\chi\in\,]0,+\infty[$, $\sigma\in\,]0,1[$, *and* $\zeta\in\,]0,1[$ *such that*

$$
(\forall n\in\mathbb{N})(\forall i\in I_n)\quad
\begin{cases}
\|e_{i,n}\|\leqslant\eta\\
\langle x_{i,c_i(n)}-a_{i,n}\mid e_{i,n}\rangle\geqslant-\sigma\|x_{i,c_i(n)}-a_{i,n}\|^2\\
\langle e_{i,n}\mid a^*_{i,n}+l^*_{i,n}\rangle\leqslant\sigma\gamma_{i,c_i(n)}\|a^*_{i,n}+l^*_{i,n}\|^2
\end{cases}
\tag{16}
$$

and that

$$
(\forall n\in\mathbb{N})(\forall k\in K_n)\quad
\begin{cases}
\|f_{k,n}\|\leqslant\chi\\
\langle l_{k,n}-b_{k,n}\mid f_{k,n}\rangle\geqslant-\zeta\|l_{k,n}-b_{k,n}\|^2\\
\langle f_{k,n}\mid b^*_{k,n}-v^*_{k,d_k(n)}\rangle\leqslant\zeta\mu_{k,d_k(n)}\|b^*_{k,n}-v^*_{k,d_k(n)}\|^2.
\end{cases}
\tag{17}
$$

Then $((x_{i,n})_{i\in I},(v^*_{k,n})_{k\in K})_{n\in\mathbb{N}}$ *converges weakly to a point in* $\mathbf{Z}$.

Proof. Denote by $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ the Hilbert direct sums of $(\mathcal{H}_i)_{i\in I}$ and $(\mathcal{G}_k)_{k\in K}$, set $\mathbf{H}=\boldsymbol{\mathcal{H}}\oplus\boldsymbol{\mathcal{G}}$, and define the operators

$$\mathbf{A}\colon \mathbf{H}\to 2^{\mathbf{H}}\colon \big((x_i)_{i\in I},(v_k^*)_{k\in K}\big)\mapsto \left(\bigtimes_{i\in I}\big(-z_i^*+A_ix_i\big)\right)\times\left(\bigtimes_{k\in K}\big(r_k+B_k^{-1}v_k^*\big)\right) \tag{18}$$

and

$$\mathbf{S}\colon \mathbf{H}\to\mathbf{H}\colon \big((x_i)_{i\in I},(v_k^*)_{k\in K}\big)\mapsto \left(\left(\sum_{k\in K}L_{k,i}^*v_k^*\right)_{i\in I},\left(-\sum_{i\in I}L_{k,i}x_i\right)_{k\in K}\right). \tag{19}$$

Using the maximal monotonicity of the operators $(A_i)_{i\in I}$ and $(B_k)_{k\in K}$, we deduce from [3, Propositions 20.22 and 20.23] that $\mathbf{A}$ is maximally monotone. In addition, we observe that $\mathbf{S}$ is a bounded linear operator with $\mathbf{S}^*=-\mathbf{S}$. At the same time, it results from (18), (19), and (2) that

$$\operatorname{zer}(\mathbf{A}+\mathbf{S})=\mathbf{Z}\neq\varnothing. \tag{20}$$

Furthermore, (15) yields

$$\big[\,(\forall i\in I)(\forall n\in\mathbb{N})\;\; a_{i,n}^*\in -z_i^*+A_ia_{i,n}\,\big]\quad\text{and}\quad\big[\,(\forall k\in K)(\forall n\in\mathbb{N})\;\; b_{k,n}\in r_k+B_k^{-1}b_{k,n}^*\,\big]. \tag{21}$$

Next, define

$$(\forall k\in K)(\forall n\in\mathbb{N})\quad \overline{\vartheta}_k(n)=\max\big\{j\in\mathbb{N}\mid j\leqslant n\;\text{ and }\;k\in K_j\big\}\quad\text{and}\quad \vartheta_k(n)=d_k\big(\overline{\vartheta}_k(n)\big), \tag{22}$$

and

$$(\forall i\in I)(\forall n\in\mathbb{N})\quad \begin{cases}\overline{\ell}_i(n)=\max\big\{j\in\mathbb{N}\mid j\leqslant n\;\text{ and }\;i\in I_j\big\},\;\;\ell_i(n)=c_i\big(\overline{\ell}_i(n)\big)\\ u_{i,n}^*=\gamma_{i,\ell_i(n)}^{-1}x_{i,\ell_i(n)}-l_{i,\overline{\ell}_i(n)}^*+\gamma_{i,\ell_i(n)}^{-1}e_{i,\overline{\ell}_i(n)}\\ w_{i,n}^*=\sum_{k\in K}L_{k,i}^*v_{k,\vartheta_k(n)}^*-l_{i,\overline{\ell}_i(n)}^*.\end{cases} \tag{23}$$

Then, for every $i\in I$ and every $n\in\mathbb{N}$, it follows from (15) and [3, Proposition 23.17(ii)] that

$$a_{i,n}=a_{i,\overline{\ell}_i(n)}=J_{\gamma_{i,\ell_i(n)}A_i}\big(\gamma_{i,\ell_i(n)}(u_{i,n}^*+z_i^*)\big)=\big(\gamma_{i,\ell_i(n)}^{-1}\mathrm{Id}-z_i^*+A_i\big)^{-1}u_{i,n}^* \tag{24}$$

and, therefore, that

$$a_{i,n}^*=a_{i,\overline{\ell}_i(n)}^*=u_{i,n}^*-\gamma_{i,\ell_i(n)}^{-1}a_{i,\overline{\ell}_i(n)}=u_{i,n}^*-\gamma_{i,\ell_i(n)}^{-1}a_{i,n}. \tag{25}$$

Likewise, for every $k\in K$ and every $n\in\mathbb{N}$, upon setting

$$\begin{cases}v_{k,n}=\mu_{k,\vartheta_k(n)}v_{k,\vartheta_k(n)}^*+l_{k,\overline{\vartheta}_k(n)}+f_{k,\overline{\vartheta}_k(n)}\\ w_{k,n}=l_{k,\overline{\vartheta}_k(n)}-\sum_{i\in I}L_{k,i}x_{i,\ell_i(n)}\end{cases} \tag{26}$$

as well as invoking (22), we get from (15) and [3, Proposition 23.17(iii)] that

$$b_{k,n}=b_{k,\overline{\vartheta}_k(n)}=J_{\mu_{k,\vartheta_k(n)}B_k(\cdot-r_k)}v_{k,n} \tag{27}$$

and, in turn, from (15) and [3, Proposition 23.20] that

$$\begin{aligned}b_{k,n}^*&=b_{k,\overline{\vartheta}_k(n)}^*\\&=\mu_{k,\vartheta_k(n)}^{-1}\big(v_{k,n}-b_{k,\overline{\vartheta}_k(n)}\big)\end{aligned} \tag{28}$$

$$
\begin{aligned}
&= \mu_{k,\vartheta_k(n)}^{-1}(v_{k,n} - b_{k,n}) && (29)\\
&= J_{\mu_{k,\vartheta_k(n)}^{-1}(r_k+B_k^{-1})}\big(\mu_{k,\vartheta_k(n)}^{-1} v_{k,n}\big)\\
&= \big(\mu_{k,\vartheta_k(n)}\mathrm{Id} + r_k + B_k^{-1}\big)^{-1} v_{k,n}. && (30)
\end{aligned}
$$

Let us set

$$
(\forall n\in\mathbb{N})\quad
\begin{cases}
\mathbf{x}_n = \big((x_{i,n})_{i\in I}, (v_{k,n}^*)_{k\in K}\big), \;\; \mathbf{u}_n = \big((x_{i,\ell_i(n)})_{i\in I}, (v_{k,\vartheta_k(n)}^*)_{k\in K}\big)\\
\mathbf{e}_n^* = \big((w_{i,n}^*)_{i\in I}, (w_{k,n})_{k\in K}\big), \;\; \mathbf{f}_n^* = \big((\gamma_{i,\ell_i(n)}^{-1} e_{i,\bar{\ell}_i(n)})_{i\in I}, (f_{k,\bar{\vartheta}_k(n)})_{k\in K}\big)\\
\mathbf{u}_n^* = \big((u_{i,n}^*)_{i\in I}, (v_{k,n})_{k\in K}\big), \;\; \mathbf{y}_n = \big((a_{i,n})_{i\in I}, (b_{k,n}^*)_{k\in K}\big)\\
\mathbf{a}_n^* = \big((a_{i,n}^*)_{i\in I}, (b_{k,n})_{k\in K}\big), \;\; \mathbf{y}_n^* = \big((t_{i,n}^*)_{i\in I}, (t_{k,n})_{k\in K}\big)\\
\mathbf{F}_n\colon \mathbf{H}\to\mathbf{H}\colon \big((x_i)_{i\in I}, (v_k^*)_{k\in K}\big) \mapsto \big((\gamma_{i,\ell_i(n)}^{-1} x_i)_{i\in I}, (\mu_{k,\vartheta_k(n)} v_k^*)_{k\in K}\big).
\end{cases}
\tag{31}
$$

Then, the operators $(\mathbf{F}_n)_{n\in\mathbb{N}}$ are ε-strongly monotone and $(1/\varepsilon)$-Lipschitzian. For every $n\in\mathbb{N}$, by virtue of (23) and (26), we deduce from (19) that

$$
\mathbf{S}\mathbf{u}_n - \mathbf{e}_n^* = \Big(\big(l_{i,\bar{\ell}_i(n)}^*\big)_{i\in I}, \big(-l_{k,\bar{\vartheta}_k(n)}\big)_{k\in K}\Big),
\tag{32}
$$

which yields

$$
\mathbf{u}_n^* = \mathbf{F}_n\mathbf{u}_n - \mathbf{S}\mathbf{u}_n + \mathbf{e}_n^* + \mathbf{f}_n^*.
\tag{33}
$$

Furthermore, we infer from (24), (30), and (18) that

$$
(\forall n\in\mathbb{N})\quad \mathbf{y}_n = (\mathbf{F}_n + \mathbf{A})^{-1}\mathbf{u}_n^*.
\tag{34}
$$

At the same time, (25) and (29) imply that

$$
(\forall n\in\mathbb{N})\quad \mathbf{a}_n^* = \mathbf{u}_n^* - \mathbf{F}_n\mathbf{y}_n,
\tag{35}
$$

while (31), (15), and (19) guarantee that

$$
(\forall n\in\mathbb{N})\quad \mathbf{y}_n^* = \mathbf{a}_n^* + \mathbf{S}\mathbf{y}_n \quad\text{and}\quad \pi_n = \langle \mathbf{x}_n \mid \mathbf{y}_n^*\rangle - \langle \mathbf{y}_n \mid \mathbf{a}_n^*\rangle.
\tag{36}
$$

Altogether, it follows from (33)–(36) that (15) is an instantiation of (3). Hence, Theorem 3(i) yields $\sum_{n\in\mathbb{N}}\|\mathbf{x}_{n+1}-\mathbf{x}_n\|^2 < +\infty$. In turn, using (13), (14), (22), and (23), we deduce from [5, Lemma A.3] that, for every $i\in I$ and every $k\in K$, we have $\mathbf{x}_{\ell_i(n)} - \mathbf{x}_n \to \mathbf{0}$ and $\mathbf{x}_{\vartheta_k(n)} - \mathbf{x}_n \to \mathbf{0}$. This and (31) imply that

$$
\mathbf{u}_n - \mathbf{x}_n \to \mathbf{0}.
\tag{37}
$$

Moreover, in view of (15), we deduce from (23) that

$$
(\forall i\in I)\quad \|w_{i,n}^*\| \leqslant \sum_{k\in K}\|L_{k,i}^*\|\,\big\|v_{k,\vartheta_k(n)}^* - v_{k,\ell_i(n)}^*\big\| \leqslant \sum_{k\in K}\|L_{k,i}^*\|\,\|\mathbf{x}_{\vartheta_k(n)} - \mathbf{x}_{\ell_i(n)}\| \to 0
\tag{38}
$$

and from (26) that

$$
(\forall k\in K)\quad \|w_{k,n}\| \leqslant \sum_{i\in I}\|L_{k,i}\|\,\|x_{i,\vartheta_k(n)} - x_{i,\ell_i(n)}\| \leqslant \sum_{i\in I}\|L_{k,i}\|\,\|\mathbf{x}_{\vartheta_k(n)} - \mathbf{x}_{\ell_i(n)}\| \to 0.
\tag{39}
$$

Therefore, $\mathbf{e}_n^* \to \mathbf{0}$. By (16) and (17), $(\mathbf{f}_n^*)_{n\in\mathbb{N}}$ is bounded. In view of (31), (16), (17), and (32), we deduce that

$$(\forall n\in\mathbb{N})\quad \langle \mathbf{u}_n-\mathbf{y}_n \mid \mathbf{f}_n^*\rangle = \sum_{i\in I}\big\langle x_{i,\ell_i(n)}-a_{i,n} \mid \gamma_{i,\ell_i(n)}^{-1}e_{i,\overline{\ell}_i(n)}\big\rangle + \sum_{k\in K}\big\langle v_{k,\vartheta_k(n)}^*-b_{k,n}^* \mid f_{k,\overline{\vartheta}_k(n)}\big\rangle$$
$$\geqslant -\sigma\sum_{i\in I}\gamma_{i,\ell_i(n)}^{-1}\|x_{i,\ell_i(n)}-a_{i,n}\|^2-\zeta\sum_{k\in K}\mu_{k,\vartheta_k(n)}\|v_{k,\vartheta_k(n)}^*-b_{k,n}^*\|^2$$
$$\geqslant -\max\{\sigma,\zeta\}\langle \mathbf{u}_n-\mathbf{y}_n \mid \mathbf{F}_n\mathbf{u}_n-\mathbf{F}_n\mathbf{y}_n\rangle \tag{40}$$

and that

$$\langle \mathbf{a}_n^*+\mathbf{S}\mathbf{u}_n-\mathbf{e}_n^* \mid \mathbf{f}_n^*\rangle = \sum_{i\in I}\big\langle a_{i,\overline{\ell}_i(n)}^*+l_{i,\overline{\ell}_i(n)}^* \mid \gamma_{i,\ell_i(n)}^{-1}e_{i,\overline{\ell}_i(n)}\big\rangle + \sum_{k\in K}\big\langle b_{k,\overline{\vartheta}_k(n)}-l_{k,\overline{\vartheta}_k(n)} \mid f_{k,\overline{\vartheta}_k(n)}\big\rangle$$
$$\leqslant \sigma\sum_{i\in I}\big\|a_{i,\overline{\ell}_i(n)}^*+l_{i,\overline{\ell}_i(n)}^*\big\|^2+\zeta\sum_{k\in K}\big\|b_{k,\overline{\vartheta}_k(n)}-l_{k,\overline{\vartheta}_k(n)}\big\|^2$$
$$\leqslant \max\{\sigma,\zeta\}\|\mathbf{a}_n^*+\mathbf{S}\mathbf{u}_n-\mathbf{e}_n^*\|^2. \tag{41}$$

Altogether, the conclusion follows from Theorem 3(ii). ☐

Remark 5 Here are a few comments on Corollary 4.

(i) Using similar arguments, one can show that the asynchronous strongly convergent block-iterative method [7, Algorithm 14] and its special case [2, Eq. (3.10)] can be viewed as instances of [4, Theorem 4.8].

(ii) In the special case of (15) where $I=\{1\}$ and

$$(\forall n\in\mathbb{N})\quad K_n=K\quad\text{and}\quad \begin{cases} e_{1,n}=0,\;\; c_1(n)=n\\ (\forall k\in K)\;\; f_{k,n}=0,\;\; d_k(n)=n, \end{cases} \tag{42}$$

the connection between [7, Theorem 13] and an instance of the warped proximal algorithm was established in [9, Proposition 19]. Nevertheless, it does not seem possible to prove [7, Theorem 13] in its full generality by using the techniques of [9].

Remark 6 Take $n\in\mathbb{N}$. Then, upon setting

$$\mathbf{H}_n=\big\{\mathbf{x}\in\mathbf{H} \mid \langle \mathbf{x}-\mathbf{y}_n \mid \mathbf{y}_n^*\rangle\leqslant 0\big\} \tag{43}$$

as well as invoking (9) and (31), we deduce that the update step

$$\begin{array}{l}
\pi_n=\sum_{i\in I}\big(\langle x_{i,n}\mid t_{i,n}^*\rangle-\langle a_{i,n}\mid a_{i,n}^*\rangle\big)+\sum_{k\in K}\big(\langle t_{k,n}\mid v_{k,n}^*\rangle-\langle b_{k,n}\mid b_{k,n}^*\rangle\big)\\
\text{if } \pi_n>0\\
\left\lfloor \begin{array}{l}\tau_n=\sum_{i\in I}\|t_{i,n}^*\|^2+\sum_{k\in K}\|t_{k,n}\|^2\\ \theta_n=\lambda_n\pi_n/\tau_n\end{array}\right.\\
\text{else}\\
\lfloor\;\theta_n=0\\
\text{for every } i\in I\\
\lfloor\;x_{i,n+1}=x_{i,n}-\theta_n t_{i,n}^*\\
\text{for every } k\in K\\
\lfloor\;v_{k,n+1}^*=v_{k,n}^*-\theta_n t_{k,n}
\end{array} \tag{44}$$

of (15) can be rewritten as

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \lambda_n(\mathrm{proj}_{\mathbf{H}_n}\mathbf{x}_n - \mathbf{x}_n), \tag{45}$$

which is the same as that of [7, Algorithm 12]; see [7, Eq. (22)]. Since $\mathbf{S}^* = -\mathbf{S}$, we derive from (36) and (31) that

$$\pi_n = \langle \mathbf{x}_n - \mathbf{y}_n \mid \mathbf{y}_n^* \rangle \quad \text{and} \quad \|\mathbf{y}_n^*\|^2 = \sum_{i\in I}\|t_{i,n}^*\|^2 + \sum_{k\in K}\|t_{k,n}\|^2, \tag{46}$$

from which we obtain the implication $\pi_n > 0 \Rightarrow \tau_n = \|\mathbf{y}_n^*\|^2 > 0$.

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