# Analysis and Numerical Solution of a Modular Convex Nash Equilibrium Problem*

Minh N. Bùi and Patrick L. Combettes

North Carolina State University, Department of Mathematics, Raleigh, NC 27695-8205, USA
mnbui@ncsu.edu and plc@math.ncsu.edu

**Abstract.** We investigate a modular convex Nash equilibrium problem involving nonsmooth functions acting on linear mixtures of strategies, as well as smooth coupling functions. An asynchronous block-iterative decomposition method is proposed to solve it.

## 1 Introduction

We consider a noncooperative game with $p$ players indexed by $I = \{1, \ldots, p\}$, in which the strategy $x_i$ of player $i \in I$ lies in a real Hilbert space $\mathcal{H}_i$. A strategy profile is a point $\boldsymbol{x} = (x_i)_{i \in I}$ in the Hilbert direct sum $\boldsymbol{\mathcal{H}} = \bigoplus_{i \in I} \mathcal{H}_i$, and the associated profile of the players other than $i \in I$ is the vector $\boldsymbol{x}_{\smallsetminus i} = (x_j)_{j \in I \smallsetminus \{i\}}$ in $\boldsymbol{\mathcal{H}}_{\smallsetminus i} = \bigoplus_{j \in I \smallsetminus \{i\}} \mathcal{H}_j$. For every $i \in I$ and every $(x_i, \boldsymbol{y}) \in \mathcal{H}_i \times \boldsymbol{\mathcal{H}}$, we set $(x_i; \boldsymbol{y}_{\smallsetminus i}) = (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_p)$. Given a real Hilbert space $\mathcal{H}$, we denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $\varphi \colon \mathcal{H} \to \left]-\infty, +\infty\right]$ which are proper in the sense that $\operatorname{dom} \varphi = \big\{ x \in \mathcal{H} \mid \varphi(x) < +\infty \big\} \neq \varnothing$.

A fundamental equilibrium notion was introduced by Nash in [30, 31] to describe a state in which the loss of each player cannot be reduced by unilateral deviation. A general formulation of the Nash equilibrium problem is

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I) \;\; x_i \in \operatorname{Argmin} \boldsymbol{\ell}_i(\cdot\,; \boldsymbol{x}_{\smallsetminus i}), \tag{1.1}$$

where $\boldsymbol{\ell}_i \colon \boldsymbol{\mathcal{H}} \to \left]-\infty, +\infty\right]$ is the global loss function of player $i \in I$. We make the following assumption: for every $i \in I$ and every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, the function $\boldsymbol{\ell}_i(\cdot\,; \boldsymbol{x}_{\smallsetminus i})$ is convex. Such convex Nash equilibrium problems have been studied since the early 1970s [7]; see [4, 8, 9, 13, 17, 20, 21, 24, 25, 28, 37] for further work. We consider the following modular formulation of (1.1), wherein the functions $(\boldsymbol{\ell}_i)_{i \in I}$ are decomposed into elementary components. This decomposition will provide more modeling flexibility and lead to efficient solution methods.

*Contact author: P. L. Combettes. Email: plc@math.ncsu.edu. Phone: +1 919 515 2671. This work was supported by the National Science Foundation under grant DMS-1818946.

**Problem 1.1** Let $(\mathcal{H}_i)_{i\in I}$ and $(\mathcal{G}_k)_{k\in K}$ be finite families of real Hilbert spaces, and set $\boldsymbol{\mathcal{H}} = \bigoplus_{i\in I}\mathcal{H}_i$ and $\boldsymbol{\mathcal{G}} = \bigoplus_{k\in K}\mathcal{G}_k$. Suppose that the following are satisfied:

[a] For every $i \in I$, $\varphi_i \in \Gamma_0(\mathcal{H}_i)$.

[b] For every $i \in I$, $\boldsymbol{f}_i\colon \boldsymbol{\mathcal{H}} \to \mathbb{R}$ is such that, for every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, $\boldsymbol{f}_i(\cdot\,;\boldsymbol{x}_{\smallsetminus i})\colon \mathcal{H}_i \to \mathbb{R}$ is convex and differentiable, and we denote its gradient at $x_i$ by $\nabla_i \boldsymbol{f}_i(\boldsymbol{x})$. Further, the operator $\boldsymbol{G}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (\nabla_i \boldsymbol{f}_i(\boldsymbol{x}))_{i\in I}$ is monotone and Lipschitzian.

[c] For every $k \in K$, $g_k \in \Gamma_0(\mathcal{G}_k)$ and $\boldsymbol{L}_k\colon \boldsymbol{\mathcal{H}} \to \mathcal{G}_k$ is linear and bounded.

The goal is to

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I)\;\; x_i \in \operatorname{Argmin} \varphi_i + \boldsymbol{f}_i(\cdot\,;\boldsymbol{x}_{\smallsetminus i}) + \sum_{k\in K}(g_k \circ \boldsymbol{L}_k)(\cdot\,;\boldsymbol{x}_{\smallsetminus i}). \tag{1.2}$$

In Problem 1.1, the individual loss of player $i \in I$ is a nondifferentiable function $\varphi_i$, while his joint loss is decomposed into a differentiable function $\boldsymbol{f}_i$ and a sum of nonsmooth functions $(g_k)_{k\in K}$ acting on linear mixtures of the strategies. To the best of our knowledge, such a general formulation of a convex Nash equilibrium has not been considered in the literature. As will be seen in Section 3, it constitutes a flexible framework that subsumes a variety of existing equilibrium models. In Section 4, we embed Problem 1.1 in an inclusion problem in the bigger space $\boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$, and we employ the new problem to provide conditions for the existence of solutions to (1.2). This embedding is also exploited in Section 5 to devise an asynchronous block-iterative algorithm to solve Problem 1.1. The proposed method features several innovations that are particularly relevant in large-scale problems: first, each function and each linear operator in (1.2) is activated separately; second, only a subgroup of functions needs to be activated at any iteration; third, the computations are asynchronous in the sense that the result of calculations initiated at earlier iterations can be incorporated at the current one.

## 2 Notation

General background on monotone operators and related notions can be found in [6]. Let $\mathcal{H}$ be a real Hilbert space. We denote by $2^{\mathcal{H}}$ the power set of $\mathcal{H}$ and by $\mathrm{Id}$ the identity operator on $\mathcal{H}$. Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$. The domain of $A$ is $\operatorname{dom} A = \{x \in \mathcal{H} \mid Ax \neq \varnothing\}$, the range of $A$ is $\operatorname{ran} A = \bigcup_{x\in\operatorname{dom} A} Ax$, the graph of $A$ is $\operatorname{gra} A = \{(x,x^*) \in \mathcal{H}\times\mathcal{H} \mid x^* \in Ax\}$, the set of zeros of $A$ is $\operatorname{zer} A = \{x \in \mathcal{H} \mid 0 \in Ax\}$, the inverse of $A$ is $A^{-1}\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x^* \mapsto \{x \in \mathcal{H} \mid x^* \in Ax\}$, and the resolvent of $A$ is $J_A = (\mathrm{Id} + A)^{-1}$. Now suppose that $A$ is monotone, that is,

$$\big(\forall (x,x^*) \in \operatorname{gra} A\big)\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle x - y \mid x^* - y^*\rangle \geqslant 0. \tag{2.1}$$

Then $A$ is maximally monotone if, for every monotone operator $\widetilde{A}\colon \mathcal{H} \to 2^{\mathcal{H}}$, $\operatorname{gra} A \subset \operatorname{gra}\widetilde{A} \Rightarrow A = \widetilde{A}$; $A$ is strongly monotone with constant $\alpha \in \left]0,+\infty\right[$ if $A - \alpha\mathrm{Id}$ is monotone; and $A$ is $3^*$ monotone if

$$(\forall x \in \operatorname{dom} A)(\forall x^* \in \operatorname{ran} A) \quad \sup_{(y,y^*)\in\operatorname{gra} A} \langle x - y \mid y^* - x^*\rangle < +\infty. \tag{2.2}$$

Let $\varphi \in \Gamma_0(\mathcal{H})$. Then $\varphi$ is supercoercive if $\lim_{\|x\|\to+\infty} \varphi(x)/\|x\| = +\infty$ and uniformly convex if there exists an increasing function $\phi\colon \left[0,+\infty\right[ \to \left[0,+\infty\right]$ that vanishes only at $0$ such that

$$(\forall x \in \operatorname{dom}\varphi)(\forall y \in \operatorname{dom}\varphi)(\forall \alpha \in \left]0,1\right[)$$
$$\varphi\big(\alpha x + (1-\alpha)y\big) + \alpha(1-\alpha)\phi\big(\|x-y\|\big) \leqslant \alpha\varphi(x) + (1-\alpha)\varphi(y). \tag{2.3}$$

For every $x \in \mathcal{H}$, $\operatorname{prox}_\varphi x$ denotes the unique minimizer of $\varphi + (1/2)\|\cdot - x\|^2$. The subdifferential of $\varphi$ is the maximally monotone operator

$$\partial\varphi\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{x^* \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \ \langle y - x \mid x^*\rangle + \varphi(x) \leqslant \varphi(y)\big\}. \tag{2.4}$$

Finally, given a nonempty convex subset $C$ of $\mathcal{H}$, the indicator function of $C$ is

$$\iota_C\colon \mathcal{H} \to [0,+\infty] : x \mapsto \begin{cases} 0, & \text{if } \ x \in C;\\ +\infty, & \text{otherwise}, \end{cases} \tag{2.5}$$

and the strong relative interior of $C$ is

$$\operatorname{sri} C = \left\{ x \in C \;\middle|\; \bigcup_{\lambda \in \left]0,+\infty\right[} \lambda(C - x) \ \text{ is a closed vector subspace of } \ \mathcal{H} \right\}. \tag{2.6}$$

## 3 Instantiations of Problem 1.1

Throughout this section, $\mathcal{H}$ is a real Hilbert space. We illustrate the wide span of Problem 1.1 by showing that common formulations encountered in various fields can be recast as special cases of it.

**Example 3.1 (quadratic coupling)** Let $I$ be a nonempty finite set. For every $i \in I$, let $\varphi_i \in \Gamma_0(\mathcal{H})$, let $\Lambda_i$ be a nonempty finite set, let $(\omega_{i,\ell,j})_{\ell\in\Lambda_i, j\in I\smallsetminus\{i\}}$ be in $\left[0,+\infty\right[$, and let $(\kappa_{i,\ell})_{\ell\in\Lambda_i}$ be in $\left]0,+\infty\right[$. Additionally, set $\boldsymbol{\mathcal{H}} = \bigoplus_{i\in I}\mathcal{H}$. The problem is to

$$\text{find } \ \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \ \text{ such that } \ (\forall i \in I) \ \ x_i \in \operatorname{Argmin} \varphi_i + \sum_{\ell\in\Lambda_i} \frac{\kappa_{i,\ell}}{2}\Bigg\| \cdot - \sum_{j\in I\smallsetminus\{i\}} \omega_{i,\ell,j} x_j \Bigg\|^2. \tag{3.1}$$

It is assumed that

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}})(\forall \boldsymbol{y} \in \boldsymbol{\mathcal{H}}) \quad \sum_{i\in I}\sum_{\ell\in\Lambda_i} \kappa_{i,\ell}\left\langle x_i - y_i \;\middle|\; x_i - y_i - \sum_{j\in I\smallsetminus\{i\}} \omega_{i,\ell,j}(x_j - y_j)\right\rangle \geqslant 0. \tag{3.2}$$

Define

$$(\forall i \in I) \quad \boldsymbol{f}_i\colon \boldsymbol{\mathcal{H}} \to \mathbb{R}\colon \boldsymbol{x} \mapsto \sum_{\ell\in\Lambda_i} \frac{\kappa_{i,\ell}}{2}\Bigg\| x_i - \sum_{j\in I\smallsetminus\{i\}} \omega_{i,\ell,j} x_j \Bigg\|^2. \tag{3.3}$$

Then, for every $i \in I$ and every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, $\boldsymbol{f}_i(\cdot\,;\boldsymbol{x}_{\smallsetminus i})$ is convex and differentiable with

$$\nabla_i \boldsymbol{f}_i(\boldsymbol{x}) = \sum_{\ell\in\Lambda_i} \kappa_{i,\ell}\Bigg( x_i - \sum_{j\in I\smallsetminus\{i\}} \omega_{i,\ell,j} x_j \Bigg). \tag{3.4}$$

Hence, in view of (3.2), the operator $\boldsymbol{G}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (\nabla_i \boldsymbol{f}_i(\boldsymbol{x}))_{i\in I}$ is monotone and Lipschitzian. Thus, (3.1) is a special case of (1.2) with $K = \varnothing$ and $(\forall i \in I)$ $\mathcal{H}_i = \mathcal{H}$. This scenario unifies models found in [1, 2, 20].

**Example 3.2** In (3.1), suppose that, for every $i \in I$, $C_i$ is a nonempty closed convex subset of $\mathcal{H}_i$, $\varphi_i = \iota_{C_i}$, $\Lambda_i = \{1\}$, and $\kappa_{i,1} = 1$. Then (3.1) becomes

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I) \;\; x_i \in \operatorname{Argmin}_{C_i} \Bigg\| \cdot - \sum_{j \in I \smallsetminus \{i\}} \omega_{i,1,j} x_j \Bigg\|^2. \tag{3.5}$$

In addition, (3.2) is satisfied when

$$\begin{cases} (\forall i \in I) \;\; \sum_{j \in I \smallsetminus \{i\}} \omega_{i,1,j} \leqslant 1 \\ (\forall j \in I) \;\; \sum_{i \in I \smallsetminus \{j\}} \omega_{i,1,j} \leqslant 1, \end{cases} \tag{3.6}$$

which places us in the setting of Example 3.1. The formulation (3.5)–(3.6) unifies models found in [5].

**Example 3.3 (minimax)** Let $I$ be a finite set and suppose that $\varnothing \neq J \subset I$. Let $(\mathcal{H}_i)_{i \in I}$ be real Hilbert spaces, and set $\boldsymbol{\mathcal{U}} = \bigoplus_{i \in I \smallsetminus J} \mathcal{H}_i$ and $\boldsymbol{\mathcal{V}} = \bigoplus_{j \in J} \mathcal{H}_j$. For every $i \in I$, let $\varphi_i \in \Gamma_0(\mathcal{H}_i)$. Further, let $\boldsymbol{\mathcal{L}} \colon \boldsymbol{\mathcal{U}} \oplus \boldsymbol{\mathcal{V}} \to \mathbb{R}$ be differentiable with a Lipschitzian gradient and such that, for every $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$ and every $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$, the functions $-\boldsymbol{\mathcal{L}}(\boldsymbol{u}, \cdot)$ and $\boldsymbol{\mathcal{L}}(\cdot, \boldsymbol{v})$ are convex. Consider the multivariate minimax problem

$$\underset{\boldsymbol{u} \in \boldsymbol{\mathcal{U}}}{\text{minimize}} \;\; \underset{\boldsymbol{v} \in \boldsymbol{\mathcal{V}}}{\text{maximize}} \;\; \sum_{i \in I \smallsetminus J} \varphi_i(u_i) + \boldsymbol{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}) - \sum_{j \in J} \varphi_j(v_j). \tag{3.7}$$

Now set $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{U}} \oplus \boldsymbol{\mathcal{V}}$ and define

$$(\forall i \in I) \quad \boldsymbol{f}_i \colon \boldsymbol{\mathcal{H}} \to \mathbb{R} \colon (\boldsymbol{u}, \boldsymbol{v}) \mapsto \begin{cases} \boldsymbol{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}), & \text{if } i \in I \smallsetminus J; \\ -\boldsymbol{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}), & \text{if } i \in J. \end{cases} \tag{3.8}$$

Then $\boldsymbol{\mathcal{H}} = \bigoplus_{i \in I} \mathcal{H}_i$ and (3.7) can be put in the form

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I) \;\; x_i \in \operatorname{Argmin} \varphi_i + \boldsymbol{f}_i(\cdot; \boldsymbol{x}_{\smallsetminus i}). \tag{3.9}$$

Let us verify Problem 1.1[b]. On the one hand, we have

$$(\forall i \in I)(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}}) \quad \nabla_i \boldsymbol{f}_i(\boldsymbol{x}) = \begin{cases} \nabla_i \boldsymbol{\mathcal{L}}(\boldsymbol{x}), & \text{if } i \in I \smallsetminus J; \\ -\nabla_i \boldsymbol{\mathcal{L}}(\boldsymbol{x}), & \text{if } i \in J. \end{cases} \tag{3.10}$$

Hence, the operator

$$\boldsymbol{G} \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}} \colon \boldsymbol{x} \mapsto \big(\nabla_i \boldsymbol{f}_i(\boldsymbol{x})\big)_{i \in I} = \Big( \big(\nabla_i \boldsymbol{\mathcal{L}}(\boldsymbol{x})\big)_{i \in I \smallsetminus J}, \big(-\nabla_j \boldsymbol{\mathcal{L}}(\boldsymbol{x})\big)_{j \in J} \Big) \tag{3.11}$$

is monotone [33, 34] and Lipschitzian. Consequently, (3.7) is an instantiation of (1.2). Special cases of (3.7) under the above assumptions can be found in [20, 27, 32, 35, 38, 39].

**Example 3.4 ("generalized" Nash equilibria)** Consider the setting of Problem 1.1 where [a] and [c] are respectively specialized to

[a'] For every $i \in I$, $\varphi_i = \iota_{C_i}$, where $C_i$ is a nonempty closed convex subset of $\mathcal{H}_i$.

[c'] $K = \{1\}$ and $g_1 = \iota_{D_1}$, where $D_1$ is a nonempty closed convex subset of $\mathcal{G}_1$.

Then (1.2) reduces to

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I) \;\; x_i \in \operatorname{Argmin}_{C_i} \boldsymbol{f}_i(\cdot\,; \boldsymbol{x}_{\smallsetminus i}) + (\iota_{D_1} \circ \boldsymbol{L}_1)(\cdot\,; \boldsymbol{x}_{\smallsetminus i}). \tag{3.12}$$

This formulation is often referred to as a generalized Nash equilibrium; see, e.g., [24, 28, 29]. However, as noted in [36], it is really a standard Nash equilibrium in the sense of (1.1) since functions are allowed to take the value $+\infty$.

**Example 3.5 (PDE model)** Let $\Omega$ be a nonempty open bounded subset of $\mathbb{R}^N$. In Example 3.4, suppose that, for every $i \in I$, $\mathcal{H}_i = L^2(\Omega)$. Let $z \in L^2(\Omega)$, let $(\Omega_i)_{i\in I}$ be nonempty open subsets of $\Omega$ with characteristic functions $(1_{\Omega_i})_{i\in I}$, and, for every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, let $\boldsymbol{S}\boldsymbol{x}$ be the unique weak solution in $H_0^1(\Omega)$ of the Dirichlet boundary value problem [23, Chapter IV.2.1]

$$\begin{cases} -\Delta y = z + \displaystyle\sum_{i\in I} 1_{\Omega_i} x_i, & \text{on } \Omega; \\ y = 0, & \text{on } \operatorname{bdry}\Omega. \end{cases} \tag{3.13}$$

For every $i \in I$, let $r_i \in \mathcal{H}_i$, let $\alpha_i \in \left]0, +\infty\right[$, and suppose that

$$\boldsymbol{f}_i \colon \boldsymbol{x} \mapsto \frac{\alpha_i}{2}\|x_i\|^2_{\mathcal{H}_i} + \frac{1}{2}\|\boldsymbol{S}\boldsymbol{x} - r_i\|^2_{\mathcal{H}_i}. \tag{3.14}$$

In addition, suppose that $\mathcal{G}_1 = H_0^1(\Omega)$ and $\boldsymbol{L}_1 = \boldsymbol{S} - \boldsymbol{S}\boldsymbol{0}$. Then we recover frameworks investigated in [9, 29].

**Example 3.6 (multivariate minimization)** Consider the setting of Problem 1.1 where [b] and [c] are respectively specialized to

[b'] For every $i \in I$, $\boldsymbol{f}_i = \boldsymbol{f}$, where $\boldsymbol{f} \colon \boldsymbol{\mathcal{H}} \to \mathbb{R}$ is a differentiable convex function such that $\boldsymbol{G} = \nabla \boldsymbol{f}$ is Lipschitzian.

[c'] For every $k \in K$, $g_k \colon \mathcal{G}_k \to \mathbb{R}$ is convex and Gâteaux differentiable, and $\boldsymbol{L}_k \colon \boldsymbol{\mathcal{H}} \to \mathcal{G}_k \colon \boldsymbol{x} \mapsto \sum_{j\in I} L_{k,j} x_j$ where, for every $j \in I$, $L_{k,j} \colon \mathcal{H}_j \to \mathcal{G}_k$ is linear and bounded.

Then (1.2) reduces to the multivariate minimization problem

$$\underset{\boldsymbol{x}\in\boldsymbol{\mathcal{H}}}{\text{minimize}} \;\; \sum_{i\in I} \varphi_i(x_i) + \boldsymbol{f}(\boldsymbol{x}) + \sum_{k\in K} g_k\left(\sum_{j\in I} L_{k,j} x_j\right). \tag{3.15}$$

Instances of this problem are found in [3, 4, 11, 12, 14, 22, 26].

## 4 Existence of solutions

Our first existence result revolves around an embedding of Problem 1.1 in a larger inclusion problem in the space $\boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$.

**Proposition 4.1** *Consider the setting of Problem 1.1 and set* $(\forall i \in I)\ \Pi_i \colon \boldsymbol{\mathcal{H}} \to \mathcal{H}_i \colon \boldsymbol{x} \mapsto x_i$. *Suppose that* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{v}}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ *satisfies*

$$\begin{cases} (\forall i \in I) \quad -\nabla_i \boldsymbol{f}_i(\overline{\boldsymbol{x}}) - \sum_{k \in K} \Pi_i(\boldsymbol{L}_k^* \overline{v}_k^*) \in \partial\varphi_i(\overline{x}_i) \\ (\forall k \in K) \ \ \boldsymbol{L}_k \overline{\boldsymbol{x}} \in \partial g_k^*(\overline{v}_k^*). \end{cases} \tag{4.1}$$

*Then* $\overline{\boldsymbol{x}}$ *solves* (1.2).

*Proof*. Take $i \in I$ and set

$$f_i = \boldsymbol{f}_i(\cdot\,; \overline{\boldsymbol{x}}_{\smallsetminus i}), \quad \overline{\boldsymbol{s}}_i = (0; \overline{\boldsymbol{x}}_{\smallsetminus i}), \quad \text{and} \quad (\forall k \in K)\ \ \widetilde{g}_k = (g_k \circ \boldsymbol{L}_k)(\cdot\,; \overline{\boldsymbol{x}}_{\smallsetminus i}). \tag{4.2}$$

Then, by Problem 1.1[b], $f_i \colon \mathcal{H}_i \to \mathbb{R}$ is convex and Gâteaux differentiable, and $\nabla f_i(\overline{x}_i) = \nabla_i \boldsymbol{f}_i(\overline{\boldsymbol{x}})$. At the same time,

$$(\forall k \in K)(\forall x_i \in \mathcal{H}_i) \quad \widetilde{g}_k(x_i) = (g_k \circ \boldsymbol{L}_k)(\Pi_i^* x_i + \overline{\boldsymbol{s}}_i) = g_k(\boldsymbol{L}_k(\Pi_i^* x_i) + \boldsymbol{L}_k \overline{\boldsymbol{s}}_i) \tag{4.3}$$

and it thus results from [6, Proposition 16.6(ii)] that

$$(\forall k \in K)(\forall x_i \in \mathcal{H}_i) \quad (\Pi_i \circ \boldsymbol{L}_k^*)\big(\partial g_k(\boldsymbol{L}_k(\Pi_i^* x_i) + \boldsymbol{L}_k \overline{\boldsymbol{s}}_i)\big) \subset \partial \widetilde{g}_k(x_i). \tag{4.4}$$

In particular,

$$(\forall k \in K) \quad (\Pi_i \circ \boldsymbol{L}_k^*)\big(\partial g_k(\boldsymbol{L}_k \overline{\boldsymbol{x}})\big) = (\Pi_i \circ \boldsymbol{L}_k^*)\Big(\partial g_k\big(\boldsymbol{L}_k(\Pi_i^* \overline{x}_i) + \boldsymbol{L}_k \overline{\boldsymbol{s}}_i\big)\Big) \subset \partial \widetilde{g}_k(\overline{x}_i). \tag{4.5}$$

Hence, we deduce from (4.1) and [6, Proposition 16.6(ii)] that

$$\begin{aligned} 0 &\in \partial\varphi_i(\overline{x}_i) + \nabla_i \boldsymbol{f}_i(\overline{\boldsymbol{x}}) + \sum_{k \in K} \Pi_i(\boldsymbol{L}_k^* \overline{v}_k^*) \\ &\subset \partial\varphi_i(\overline{x}_i) + \nabla f_i(\overline{x}_i) + \sum_{k \in K} (\Pi_i \circ \boldsymbol{L}_k^*)\big(\partial g_k(\boldsymbol{L}_k \overline{\boldsymbol{x}})\big) \\ &\subset \partial\varphi_i(\overline{x}_i) + \nabla f_i(\overline{x}_i) + \sum_{k \in K} \partial \widetilde{g}_k(\overline{x}_i) \\ &\subset \partial\Bigg(\varphi_i + f_i + \sum_{k \in K} \widetilde{g}_k\Bigg)(\overline{x}_i). \end{aligned} \tag{4.6}$$

Consequently, appealing to Fermat's rule [6, Theorem 16.3] and (4.2), we arrive at

$$\overline{x}_i \in \text{Argmin}\, \varphi_i + \boldsymbol{f}_i(\cdot\,; \overline{\boldsymbol{x}}_{\smallsetminus i}) + \sum_{k \in K} (g_k \circ \boldsymbol{L}_k)(\cdot\,; \overline{\boldsymbol{x}}_{\smallsetminus i}), \tag{4.7}$$

which completes the proof. ☐

We are now in a position to provide specific existence conditions.

**Proposition 4.2** *Consider the setting of Problem 1.1, set*

$$\boldsymbol{C} = \big\{ (\boldsymbol{L}_k \boldsymbol{x} - y_k)_{k \in K} \mid (\forall i \in I) \ \ x_i \in \text{dom}\, \varphi_i \ \ \textit{and} \ \ (\forall k \in K) \ \ y_k \in \text{dom}\, g_k \big\}, \tag{4.8}$$

*and let* $\boldsymbol{Z} \subset \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ *be the set of solutions to* (4.1). *Suppose that* $\boldsymbol{0} \in \text{sri}\, \boldsymbol{C}$ *and that one of the following is satisfied:*

(i) *For every $i \in I$, one of the following holds:*

1/ $\partial\varphi_i$ *is surjective.*

2/ $\varphi_i$ *is supercoercive.*

3/ $\operatorname{dom}\varphi_i$ *is bounded.*

4/ $\varphi_i$ *is uniformly convex.*

(ii) $\boldsymbol{G}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (\nabla_i \boldsymbol{f}_i(\boldsymbol{x}))_{i\in I}$ *is* $3^*$ *monotone and surjective.*

*Then $\boldsymbol{Z} \neq \varnothing$ and Problem 1.1 has a solution.*

*Proof.* Define

$$\begin{cases} \boldsymbol{A}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}\colon \boldsymbol{x} \mapsto \times_{i\in I} \partial\varphi_i(x_i) \\ \boldsymbol{B}\colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}}\colon \boldsymbol{y} \mapsto \times_{k\in K} \partial g_k(y_k) \\ \boldsymbol{L}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}}\colon \boldsymbol{x} \mapsto (\boldsymbol{L}_k\boldsymbol{x})_{k\in K} \end{cases} \tag{4.9}$$

and

$$\boldsymbol{T}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}\colon \boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} + \boldsymbol{L}^*\big(\boldsymbol{B}(\boldsymbol{L}\boldsymbol{x})\big) + \boldsymbol{G}\boldsymbol{x}. \tag{4.10}$$

Note that the adjoint of $\boldsymbol{L}$ is

$$\boldsymbol{L}^*\colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{v}^* \mapsto \sum_{k\in K} \boldsymbol{L}_k^* v_k^*. \tag{4.11}$$

Now suppose that $\overline{\boldsymbol{x}} \in \operatorname{zer}\boldsymbol{T}$. Then there exists $\overline{\boldsymbol{v}}^* \in \boldsymbol{B}(\boldsymbol{L}\overline{\boldsymbol{x}})$ such that $-\boldsymbol{G}\overline{\boldsymbol{x}} - \boldsymbol{L}^*\overline{\boldsymbol{v}}^* \in \boldsymbol{A}\overline{\boldsymbol{x}}$ or, equivalently, by Problem 1.1[b] and (4.11), $(\forall i \in I)\ -\nabla_i \boldsymbol{f}_i(\overline{\boldsymbol{x}}) - \sum_{k\in K} \Pi_i(\boldsymbol{L}_k^*\overline{v}_k^*) \in \partial\varphi_i(\overline{x}_i)$. Further, (4.9) yields $\overline{v}_k^* \in \partial g_k(\boldsymbol{L}_k\overline{\boldsymbol{x}})$. Altogether, in view of (4.1) and Proposition 4.1, we have established the implications

$$\operatorname{zer}\boldsymbol{T} \neq \varnothing \quad \Rightarrow \quad \boldsymbol{Z} \neq \varnothing \quad \Rightarrow \quad \text{Problem 1.1 has a solution.} \tag{4.12}$$

Therefore, it suffices to show that $\operatorname{zer}\boldsymbol{T} \neq \varnothing$. To do so, define

$$\begin{cases} \boldsymbol{\varphi}\colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right]\colon \boldsymbol{x} \mapsto \sum_{i\in I} \varphi_i(x_i) \\ \boldsymbol{g}\colon \boldsymbol{\mathcal{G}} \to \left]-\infty,+\infty\right]\colon \boldsymbol{y} \mapsto \sum_{k\in K} g_k(y_k) \\ \boldsymbol{Q} = \boldsymbol{A} + \boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}. \end{cases} \tag{4.13}$$

Then, by (4.9) and [6, Proposition 16.9], $\boldsymbol{A} = \partial\boldsymbol{\varphi}$ and $\boldsymbol{B} = \partial\boldsymbol{g}$. In turn, since (4.8) and (4.9) imply that $\boldsymbol{0} \in \operatorname{sri}\boldsymbol{C} = \operatorname{sri}(\boldsymbol{L}(\operatorname{dom}\boldsymbol{\varphi}) - \operatorname{dom}\boldsymbol{g})$, we derive from [6, Theorem 16.47(i)] that $\boldsymbol{Q} = \partial(\boldsymbol{\varphi} + \boldsymbol{g} \circ \boldsymbol{L})$. Therefore, in view of [6, Theorem 20.25 and Example 25.13],

$$\boldsymbol{A},\ \boldsymbol{B}, \text{ and } \boldsymbol{Q} \text{ are maximally monotone and } 3^* \text{ monotone.} \tag{4.14}$$

(i): Fix temporarily $i \in I$. By [6, Theorem 20.25], $\partial\varphi_i$ is maximally monotone. First, if (i)2/ holds, then [6, Corollary 16.30, and Propositions 14.15 and 16.27] entail that $\operatorname{ran}\partial\varphi_i = \operatorname{dom}\partial\varphi_i^* = \mathcal{H}_i$ and, hence, (i)1/ holds. Second, if (i)3/ holds, then $\operatorname{dom}\partial\varphi_i \subset \operatorname{dom}\varphi_i$ is bounded and, therefore, it follows from [6, Corollary 21.25] that (i)1/ holds. Finally, if (i)4/ holds, then [6, Proposition 17.26(ii)] implies that (i)2/ holds and, in turn, that (i)1/ holds. Altogether, it is enough to show that

$$\big[\,(\forall i \in I)\;\; \partial\varphi_i \;\text{ is surjective}\,\big] \quad\Rightarrow\quad \operatorname{zer}\boldsymbol{T} \neq \varnothing. \tag{4.15}$$

Assume that the operators $(\partial\varphi_i)_{i\in I}$ are surjective and set

$$\boldsymbol{P} = -\boldsymbol{Q}^{-1} \circ (-\mathbf{Id}) + \boldsymbol{G}^{-1}. \tag{4.16}$$

Then we derive from (4.9) that $\boldsymbol{A}$ is surjective. On the other hand, [10, Proposition 6] asserts that $\boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}$ is $3^*$ monotone. Hence, (4.14) and [6, Corollary 25.27(i)] yields

$$\operatorname{dom}\boldsymbol{Q}^{-1} = \operatorname{ran}\boldsymbol{Q} = \boldsymbol{\mathcal{H}}. \tag{4.17}$$

In turn, since $\boldsymbol{Q}^{-1}$ and $\boldsymbol{G}^{-1}$ are maximally monotone, [6, Theorem 25.3] implies that $\boldsymbol{P}$ is likewise. Furthermore, we observe that

$$\operatorname{dom}\boldsymbol{G}^{-1} \subset \boldsymbol{\mathcal{H}} = \operatorname{dom}\big(-\boldsymbol{Q}^{-1} \circ (-\mathbf{Id})\big) \tag{4.18}$$

and, by virtue of (4.14) and [6, Proposition 25.19(i)], that $-\boldsymbol{Q}^{-1} \circ (-\mathbf{Id})$ is $3^*$ monotone. Therefore, since $\operatorname{ran}\boldsymbol{G}^{-1} = \operatorname{dom}\boldsymbol{G} = \boldsymbol{\mathcal{H}}$, [6, Corollary 25.27(ii)] entails that $\boldsymbol{P}$ is surjective and, in turn, that $\operatorname{zer}\boldsymbol{P} \neq \varnothing$. Consequently, [6, Proposition 26.33(iii)] asserts that $\operatorname{zer}\boldsymbol{T} \neq \varnothing$.

(ii): Since $\boldsymbol{G}$ is maximally monotone and $\operatorname{dom}\boldsymbol{G} = \boldsymbol{\mathcal{H}}$, it results from (4.14) and [6, Theorem 25.3] that $\boldsymbol{T} = \boldsymbol{Q} + \boldsymbol{G}$ is maximally monotone. Hence, since $\boldsymbol{G}$ is surjective, we derive from (4.14) and [6, Corollary 25.27(i)] that $\boldsymbol{T}$ is surjective and, therefore, that $\operatorname{zer}\boldsymbol{T} \neq \varnothing$. ☐

**Remark 4.3** Sufficient conditions for $\mathbf{0} \in \operatorname{sri}\boldsymbol{C}$ to hold in Proposition 4.2 can be found in [18, Proposition 5.3].

## 5 Algorithm

The main result of this section is the following theorem, where we introduce an asynchronous block-iterative algorithm to solve Problem 1.1 and prove its convergence.

**Theorem 5.1** *Consider the setting of Problem 1.1 and set* $(\forall i \in I)\ \Pi_i \colon \boldsymbol{\mathcal{H}} \to \mathcal{H}_i \colon \boldsymbol{x} \mapsto x_i$. *Let* $(\chi_i)_{i\in I}$ *be a family in* $[0, +\infty[$ *such that*

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}})(\forall \boldsymbol{y} \in \boldsymbol{\mathcal{H}}) \quad \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{G}\boldsymbol{x} - \boldsymbol{G}\boldsymbol{y}\rangle \leqslant \sum_{i\in I} \chi_i \|x_i - y_i\|^2, \tag{5.1}$$

*let* $\alpha \in \left]0, +\infty\right[$ *and* $\varepsilon \in \left]0, 1\right[$ *be such that* $1/\varepsilon > \alpha + \max_{i\in I}\chi_i$, *let* $(\lambda_n)_{n\in\mathbb{N}}$ *be in* $[\varepsilon, 2-\varepsilon]$, *and let* $D \in \mathbb{N}$. *Suppose that the following are satisfied:*

[a] *There exists* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{v}}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ *such that* (4.1) *holds.*

[b] *For every* $i \in I$, $x_{i,0} \in \mathcal{H}_i$ *and, for every* $n \in \mathbb{N}$, $\gamma_{i,n} \in [\varepsilon, 1/(\chi_i + \alpha)]$ *and* $c_i(n) \in \mathbb{N}$ *satisfies* $n - D \leqslant c_i(n) \leqslant n$.

[c] *For every* $k \in K$, $v^*_{k,0} \in \mathcal{G}_k$ *and, for every* $n \in \mathbb{N}$, $\mu_{k,n} \in [\alpha, 1/\varepsilon]$ *and* $d_k(n) \in \mathbb{N}$ *satisfies* $n - D \leqslant d_k(n) \leqslant n$.

[d] *$(I_n)_{n\in\mathbb{N}}$ are nonempty subsets of $I$ and $(K_n)_{n\in\mathbb{N}}$ are nonempty subsets of $K$ such that*

$$I_0 = I, \quad K_0 = K, \quad \textit{and} \quad (\exists\, m \in \mathbb{N})(\forall n \in \mathbb{N}) \bigcup_{j=n}^{n+m} I_j = I \ \textit{and} \ \bigcup_{j=n}^{n+m} K_j = K. \tag{5.2}$$

*Further, set $\boldsymbol{L}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}}\colon \boldsymbol{x} \mapsto (\boldsymbol{L}_k\boldsymbol{x})_{k\in K}$. Iterate*

$$
\begin{array}{l}
\textit{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I_n\\
\left\lfloor
\begin{array}{l}
x_{i,n}^* = x_{i,c_i(n)} - \gamma_{i,c_i(n)}\Big(\nabla_i \boldsymbol{f}_i(\boldsymbol{x}_{c_i(n)}) + \sum_{k\in K} \Pi_i\big(\boldsymbol{L}_k^* v_{k,c_i(n)}^*\big)\Big)\\
a_{i,n} = \operatorname{prox}_{\gamma_{i,c_i(n)}\varphi_i} x_{i,n}^*\\
a_{i,n}^* = \gamma_{i,c_i(n)}^{-1}(x_{i,n}^* - a_{i,n})
\end{array}
\right.\\
\textit{for every } i \in I \smallsetminus I_n\\
\left\lfloor\ (a_{i,n}, a_{i,n}^*) = (a_{i,n-1}, a_{i,n-1}^*)\right.\\
\textit{for every } k \in K_n\\
\left\lfloor
\begin{array}{l}
y_{k,n}^* = \mu_{k,d_k(n)} v_{k,d_k(n)}^* + \boldsymbol{L}_k \boldsymbol{x}_{d_k(n)}\\
b_{k,n} = \operatorname{prox}_{\mu_{k,d_k(n)} g_k} y_{k,n}^*\\
b_{k,n}^* = \mu_{k,d_k(n)}^{-1}(y_{k,n}^* - b_{k,n})
\end{array}
\right.\\
\textit{for every } k \in K \smallsetminus K_n\\
\left\lfloor\ (b_{k,n}, b_{k,n}^*) = (b_{k,n-1}, b_{k,n-1}^*)\right.\\
\boldsymbol{t}_n^* = \boldsymbol{a}_n^* + \boldsymbol{G}\boldsymbol{a}_n + \boldsymbol{L}^*\boldsymbol{b}_n^*\\
\boldsymbol{t}_n = \boldsymbol{b}_n - \boldsymbol{L}\boldsymbol{a}_n\\
\pi_n = \langle \boldsymbol{a}_n - \boldsymbol{x}_n \mid \boldsymbol{t}_n^*\rangle + \langle \boldsymbol{t}_n \mid \boldsymbol{b}_n^* - \boldsymbol{v}_n^*\rangle\\
\textit{if } \pi_n < 0\\
\left\lfloor
\begin{array}{l}
\alpha_n = \lambda_n \pi_n/\big(\|\boldsymbol{t}_n\|^2 + \|\boldsymbol{t}_n^*\|^2\big)\\
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \alpha_n \boldsymbol{t}_n^*\\
\boldsymbol{v}_{n+1}^* = \boldsymbol{v}_n^* + \alpha_n \boldsymbol{t}_n
\end{array}
\right.\\
\textit{else}\\
\left\lfloor\ \big(\boldsymbol{x}_{n+1}, \boldsymbol{v}_{n+1}^*\big) = (\boldsymbol{x}_n, \boldsymbol{v}_n^*).\right.
\end{array}
\right.
\end{array}
\tag{5.3}
$$

*Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges weakly to a solution to Problem 1.1.*

The salient features of the proposed algorithm are the following:

- **Decomposition:** In (5.3), the functions $(\varphi_i)_{i\in I}$ and $(g_k)_{k\in K}$ are activated separately via their proximity operators.
- **Block-iterative implementation:** At iteration $n$, we require that only the subfamilies of functions $(\varphi_i)_{i\in I_n}$ and $(g_k)_{k\in K_n}$ be activated, as opposed to all of them as in standard splitting methods. To guarantee convergence, we ask in condition [d] of Theorem 5.1 that each of these functions be activated frequently enough.
- **Asynchronous implementation:** Given $i \in I$ and $k \in K$, the asynchronous character of the algorithm is materialized by the variables $c_i(n)$ and $d_k(n)$ which signal when the underlying computations incorporated at iteration $n$ were initiated. Conditions [b] and [c] of Theorem 5.1 ask that the lag between the initiation and the incorporation of such computations do not exceed

$D$ iterations. The synchronous implementation is obtained when $c_i(n) = n$ and $d_k(n) = n$ in (5.3). The introduction of asynchronous and block-iterative techniques in monotone operator splitting were initiated in [19].

In order to prove Theorem 5.1, we need to establish some preliminary properties.

**Proposition 5.2** *Let $(\mathcal{X}_i)_{i\in\mathbb{I}}$ be a finite family of real Hilbert spaces with Hilbert direct sum $\boldsymbol{\mathcal{X}} = \bigoplus_{i\in\mathbb{I}} \mathcal{X}_i$. For every $i \in \mathbb{I}$, let $P_i\colon \mathcal{X}_i \to 2^{\mathcal{X}_i}$ be maximally monotone and let $Q_i\colon \boldsymbol{\mathcal{X}} \to \mathcal{X}_i$. It is assumed that $\boldsymbol{Q}\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}\colon \boldsymbol{x} \mapsto (Q_i\boldsymbol{x})_{i\in\mathbb{I}}$ is monotone and Lipschitzian, and that the problem*

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \text{ such that } (\forall i \in \mathbb{I})\;\; 0 \in P_i x_i + Q_i\boldsymbol{x} \tag{5.4}$$

*has a solution. Let $(\chi_i)_{i\in\mathbb{I}}$ be a family in $[0,+\infty[$ such that*

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}})(\forall \boldsymbol{y} \in \boldsymbol{\mathcal{X}}) \quad \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{Q}\boldsymbol{x} - \boldsymbol{Q}\boldsymbol{y}\rangle \leqslant \sum_{i\in\mathbb{I}} \chi_i \|x_i - y_i\|^2, \tag{5.5}$$

*let $\alpha \in \left]0,+\infty\right[$, let $\varepsilon \in \left]0,1\right[$ be such that $1/\varepsilon > \alpha + \max_{i\in\mathbb{I}} \chi_i$, and let $D \in \mathbb{N}$. For every $i \in \mathbb{I}$, let $x_{i,0} \in \mathcal{X}_i$ and, for every $n \in \mathbb{N}$, let $\gamma_{i,n} \in [\varepsilon, 1/(\chi_i + \alpha)]$, let $\lambda_n \in [\varepsilon, 2-\varepsilon]$, and let $d_i(n) \in \mathbb{N}$ be such that*

$$n - D \leqslant d_i(n) \leqslant n. \tag{5.6}$$

*In addition, let $(\mathbb{I}_n)_{n\in\mathbb{N}}$ be nonempty subsets of $\mathbb{I}$ such that*

$$\mathbb{I}_0 = \mathbb{I} \quad \text{and} \quad (\exists\, m \in \mathbb{N})(\forall n \in \mathbb{N}) \;\; \bigcup_{j=n}^{n+m} \mathbb{I}_j = \mathbb{I}. \tag{5.7}$$

*Iterate*

$$\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for every } i \in \mathbb{I}_n\\
\left\lfloor
\begin{array}{l}
x_{i,n}^* = x_{i,d_i(n)} - \gamma_{i,d_i(n)} Q_i \boldsymbol{x}_{d_i(n)}\\
p_{i,n} = J_{\gamma_{i,d_i(n)} P_i} x_{i,n}^*\\
p_{i,n}^* = \gamma_{i,d_i(n)}^{-1}(x_{i,n}^* - p_{i,n})
\end{array}
\right.\\
\text{for every } i \in \mathbb{I} \smallsetminus \mathbb{I}_n\\
\left\lfloor\; (p_{i,n}, p_{i,n}^*) = (p_{i,n-1}, p_{i,n-1}^*) \right.\\
\boldsymbol{s}_n^* = \boldsymbol{p}_n^* + \boldsymbol{Q}\boldsymbol{p}_n\\
\pi_n = \langle \boldsymbol{p}_n - \boldsymbol{x}_n \mid \boldsymbol{s}_n^*\rangle\\
\text{if } \pi_n < 0\\
\left\lfloor
\begin{array}{l}
\alpha_n = \lambda_n \pi_n / \|\boldsymbol{s}_n^*\|^2\\
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \alpha_n \boldsymbol{s}_n^*
\end{array}
\right.\\
\text{else}\\
\left\lfloor\; \boldsymbol{x}_{n+1} = \boldsymbol{x}_n. \right.
\end{array}
\right.
\end{array} \tag{5.8}$$

*Then the following hold:*

(i) $(\forall i \in \mathbb{I})\; x_{i,n} - p_{i,n} \to 0.$

(ii) $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly to a solution to* (5.4).

*Proof*. Define

$$\boldsymbol{M}\colon \boldsymbol{\mathcal{X}} \to 2^{\boldsymbol{\mathcal{X}}}\colon \boldsymbol{x} \mapsto \boldsymbol{Q}\boldsymbol{x} + \bigtimes_{i\in\mathbb{I}} P_i x_i. \tag{5.9}$$

It follows from [6, Proposition 20.23] that the operator $\boldsymbol{x} \mapsto \bigtimes_{i\in\mathbb{I}} P_i x_i$ is maximally monotone. Thus, since $\boldsymbol{Q}$ is maximally monotone by [6, Corollary 20.28], we deduce from [6, Corollary 25.5(i)] that $\boldsymbol{M}$ is maximally monotone. Further, since (5.4) has a solution, $\operatorname{zer}\boldsymbol{M} \neq \varnothing$. Set

$$(\forall i \in \mathbb{I})(\forall n \in \mathbb{N}) \quad \overline{\delta}_i(n) = \max\big\{j \in \mathbb{N} \mid j \leqslant n \;\text{ and }\; i \in \mathbb{I}_j\big\} \quad \text{and} \quad \delta_i(n) = d_i\big(\overline{\delta}_i(n)\big), \tag{5.10}$$

and define

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{K}_n\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}\colon \boldsymbol{x} \mapsto \big(\gamma_{i,\delta_i(n)}^{-1} x_i\big)_{i\in\mathbb{I}} - \boldsymbol{Q}\boldsymbol{x}. \tag{5.11}$$

In addition, let $\chi$ be a Lipschitz constant of $\boldsymbol{Q}$. Then, the operators $(\boldsymbol{K}_n)_{n\in\mathbb{N}}$ are Lipschitzian with constant $\beta = \sqrt{2(\varepsilon^{-2} + \chi^2)}$. At the same time, for every $n \in \mathbb{N}$, we derive from (5.11) and (5.5) that

$$\begin{aligned}
(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}})(\forall \boldsymbol{y} \in \boldsymbol{\mathcal{X}}) \quad \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{K}_n\boldsymbol{x} - \boldsymbol{K}_n\boldsymbol{y}\rangle &= \sum_{i\in\mathbb{I}} \gamma_{i,\delta_i(n)}^{-1} \|x_i - y_i\|^2 - \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{Q}\boldsymbol{x} - \boldsymbol{Q}\boldsymbol{y}\rangle \\
&\geqslant \sum_{i\in\mathbb{I}} (\chi_i + \alpha)\|x_i - y_i\|^2 - \sum_{i\in\mathbb{I}} \chi_i \|x_i - y_i\|^2 \\
&= \alpha\|\boldsymbol{x} - \boldsymbol{y}\|^2
\end{aligned} \tag{5.12}$$

and, in turn, that $\boldsymbol{K}_n$ is $\alpha$-strongly monotone and maximally monotone [6, Corollary 20.28]. Hence, for every $n \in \mathbb{N}$, [6, Proposition 22.11(ii)] implies that there exists $\widetilde{\boldsymbol{x}}_n \in \boldsymbol{\mathcal{X}}$ such that

$$\big(\gamma_{i,\delta_i(n)}^{-1} x_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} = \boldsymbol{K}_n \widetilde{\boldsymbol{x}}_n. \tag{5.13}$$

Therefore, we infer from (5.8), (5.10), (5.9), and (5.11) that

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad \boldsymbol{p}_n &= \big(p_{i,\overline{\delta}_i(n)}\big)_{i\in\mathbb{I}} && (5.14)\\
&= \big(J_{\gamma_{i,\delta_i(n)} P_i} x_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} \\
&= (\boldsymbol{K}_n + \boldsymbol{M})^{-1}\big(\gamma_{i,\delta_i(n)}^{-1} x_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} \\
&= (\boldsymbol{K}_n + \boldsymbol{M})^{-1}\big(\boldsymbol{K}_n \widetilde{\boldsymbol{x}}_n\big). && (5.15)
\end{aligned}$$

On the other hand, by (5.8), (5.10), (5.13), (5.14), and (5.11),

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad \boldsymbol{s}_n^* &= \boldsymbol{p}_n^* + \boldsymbol{Q}\boldsymbol{p}_n \\
&= \big(p_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} + \boldsymbol{Q}\boldsymbol{p}_n \\
&= \big(\gamma_{i,\delta_i(n)}^{-1}\big(x_{i,\overline{\delta}_i(n)}^* - p_{i,\overline{\delta}_i(n)}\big)\big)_{i\in\mathbb{I}} + \boldsymbol{Q}\boldsymbol{p}_n \\
&= \big(\gamma_{i,\delta_i(n)}^{-1} x_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} - \big(\gamma_{i,\delta_i(n)}^{-1} p_{i,\overline{\delta}_i(n)}\big)_{i\in\mathbb{I}} + \boldsymbol{Q}\boldsymbol{p}_n \\
&= \boldsymbol{K}_n \widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n \boldsymbol{p}_n.
\end{aligned} \tag{5.16}$$

Thus, (5.8) can be recast as

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots \\
\left\lfloor
\begin{array}{l}
\boldsymbol{p}_n = (\boldsymbol{K}_n + \boldsymbol{M})^{-1}(\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n) \\
\boldsymbol{s}_n^* = \boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{p}_n \\
\text{if } \langle \boldsymbol{p}_n - \boldsymbol{x}_n \mid \boldsymbol{s}_n^* \rangle < 0 \\
\left\lfloor
\begin{array}{l}
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \dfrac{\lambda_n \langle \boldsymbol{p}_n - \boldsymbol{x}_n \mid \boldsymbol{s}_n^* \rangle}{\|\boldsymbol{s}_n^*\|^2}\, \boldsymbol{s}_n^*
\end{array}
\right. \\
\text{else} \\
\left\lfloor\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n. \right.
\end{array}
\right.
\end{array}
\tag{5.17}
$$

Therefore, [15, Theorem 4.2(i)] yields $\sum_{n\in\mathbb{N}} \|\boldsymbol{x}_{n+1} - \boldsymbol{x}_n\|^2 < +\infty$. On the one hand, in view of [16, Lemma A.3], we deduce from (5.7) and (5.10) that $(\forall i \in \mathbb{I})\ \boldsymbol{x}_{\delta_i(n)} - \boldsymbol{x}_n \to \boldsymbol{0}$. On the other hand, for every $n \in \mathbb{N}$, every $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$, and every $\boldsymbol{y} \in \boldsymbol{\mathcal{X}}$, we deduce from (5.12) and the Cauchy–Schwarz inequality that $\alpha\|\boldsymbol{x} - \boldsymbol{y}\|^2 \leqslant \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{K}_n\boldsymbol{x} - \boldsymbol{K}_n\boldsymbol{y}\rangle \leqslant \|\boldsymbol{x} - \boldsymbol{y}\|\,\|\boldsymbol{K}_n\boldsymbol{x} - \boldsymbol{K}_n\boldsymbol{y}\|$, from which it follows that

$$
\alpha\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant \|\boldsymbol{K}_n\boldsymbol{x} - \boldsymbol{K}_n\boldsymbol{y}\|. \tag{5.18}
$$

Hence, using (5.13), (5.8), (5.11), and the fact that $\boldsymbol{Q}$ is $\chi$-Lipschitzian, we get

$$
\begin{aligned}
\alpha^2\|\widetilde{\boldsymbol{x}}_n - \boldsymbol{x}_n\|^2 &\leqslant \|\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{x}_n\|^2 \\
&= \Big\|\big(\gamma_{i,\delta_i(n)}^{-1}\big(x_{i,\delta_i(n)} - \gamma_{i,\delta_i(n)}Q_i\boldsymbol{x}_{\delta_i(n)}\big)\big)_{i\in\mathbb{I}} - \big(\gamma_{i,\delta_i(n)}^{-1}x_{i,n} - Q_i\boldsymbol{x}_n\big)_{i\in\mathbb{I}}\Big\|^2 \\
&= \sum_{i\in\mathbb{I}} \big\|\gamma_{i,\delta_i(n)}^{-1}\big(x_{i,\delta_i(n)} - x_{i,n}\big) + \big(Q_i\boldsymbol{x}_n - Q_i\boldsymbol{x}_{\delta_i(n)}\big)\big\|^2 \\
&\leqslant \sum_{i\in\mathbb{I}} 2\big(\varepsilon^{-2}\|x_{i,\delta_i(n)} - x_{i,n}\|^2 + \|Q_i\boldsymbol{x}_n - Q_i\boldsymbol{x}_{\delta_i(n)}\|^2\big) \\
&\leqslant \sum_{i\in\mathbb{I}} 2(\varepsilon^{-2} + \chi^2)\|\boldsymbol{x}_{\delta_i(n)} - \boldsymbol{x}_n\|^2 \\
&\to 0.
\end{aligned}
\tag{5.19}
$$

Thus, we conclude via [15, Theorem 4.2(ii) and Remark 4.3] that $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges weakly to a point in zer $\boldsymbol{M}$, i.e., a solution to (5.4). Further, it is shown in the proof of [15, Theorem 4.2(ii)] that $\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{p}_n \to \boldsymbol{0}$. Hence, we derive from (5.18) and (5.19) that $\|\boldsymbol{x}_n - \boldsymbol{p}_n\| \leqslant \|\boldsymbol{x}_n - \widetilde{\boldsymbol{x}}_n\| + \|\widetilde{\boldsymbol{x}}_n - \boldsymbol{p}_n\| \leqslant \|\boldsymbol{x}_n - \widetilde{\boldsymbol{x}}_n\| + (1/\alpha)\|\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{p}_n\| \to 0$. ☐

We are now ready to prove Theorem 5.1.

*Proof*. Consider the system of monotone inclusions

$$
\text{find } (\boldsymbol{x}, \boldsymbol{v}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}} \text{ such that }
\begin{cases}
(\forall i \in I)\ \ 0 \in \partial\varphi_i(x_i) + \nabla_i \boldsymbol{f}_i(\boldsymbol{x}) + \sum_{k\in K} \Pi_i(\boldsymbol{L}_k^* v_k^*) \\
(\forall k \in K)\ \ 0 \in \partial g_k^*(v_k^*) - \boldsymbol{L}_k\boldsymbol{x}.
\end{cases}
\tag{5.20}
$$

We assume, without loss of generality, that $I$ and $K$ are disjoint subsets of $\mathbb{N}$. Then, in view of (4.11), (5.20) is a special case of (5.4) where $\mathbb{I} = I \cup K$ and

$$
\begin{cases}
(\forall i \in I)\ \ \mathcal{X}_i = \mathcal{H}_i \text{ and } P_i = \partial\varphi_i \\
(\forall k \in K)\ \ \mathcal{X}_k = \mathcal{G}_k \text{ and } P_k = \partial g_k^* \\
\boldsymbol{Q}\colon (\boldsymbol{x}, \boldsymbol{v}^*) \mapsto (\boldsymbol{G}\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{v}^*, -\boldsymbol{L}\boldsymbol{x}).
\end{cases}
\tag{5.21}
$$

Note that $\boldsymbol{Q}$ is Lipschitzian and that, for every $(\boldsymbol{x}, \boldsymbol{v}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ and every $(\boldsymbol{y}, \boldsymbol{w}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$, it follows from (5.1) that

$$\big\langle (\boldsymbol{x}, \boldsymbol{v}^*) - (\boldsymbol{y}, \boldsymbol{w}^*) \mid \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{v}^*) - \boldsymbol{Q}(\boldsymbol{y}, \boldsymbol{w}^*) \big\rangle = \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{G}\boldsymbol{x} - \boldsymbol{G}\boldsymbol{y} \rangle \leqslant \sum_{i \in I} \chi_i \|x_i - y_i\|^2. \tag{5.22}$$

In addition, for every $n \in \mathbb{N}$ and every $k \in K_n$, upon setting $z_{k,n}^* = y_{k,n}^* / \mu_{k,d_k(n)}$, we deduce from (5.3) that

$$z_{k,n}^* = v_{k,d_k(n)}^* + \mu_{k,d_k(n)}^{-1} \boldsymbol{L}_k \boldsymbol{x}_{d_k(n)} \tag{5.23}$$

and from [6, Theorem 14.3(ii) and Example 23.3] that

$$b_{k,n}^* = \operatorname{prox}_{\mu_{k,d_k(n)}^{-1} g_k^*} z_{k,n}^* = J_{\mu_{k,d_k(n)}^{-1} P_k} z_{k,n}^* \quad \text{and} \quad b_{k,n} = \mu_{k,d_k(n)} (z_{k,n}^* - b_{k,n}^*). \tag{5.24}$$

Hence, (5.3) is a realization of (5.8) in the context of (5.21) with

$$\big[\, (\forall n \in \mathbb{N})\;\; \mathbb{I}_n = I_n \cup K_n \,\big] \quad \text{and} \quad \big[\, (\forall k \in K)\;\; \chi_k = 0 \;\; \text{and} \;\; \gamma_{k,n} = \mu_{k,n}^{-1} \,\big]. \tag{5.25}$$

Moreover, we observe that $\varnothing \neq \boldsymbol{Z}$ is the set of solutions to (5.20). Hence, Proposition 5.2(ii) implies that $(\boldsymbol{x}_n, \boldsymbol{v}_n^*)_{n \in \mathbb{N}}$ converges weakly to a point $(\boldsymbol{x}, \boldsymbol{v}^*) \in \boldsymbol{Z}$. By Proposition 4.1, $\boldsymbol{x}$ solves (1.2). ☐

**Remark 5.3** By invoking [15, Theorem 4.8] and arguing as in the proof of Proposition 5.2, we obtain a strongly convergent counterpart of Proposition 5.2 which, in turn, yields a strongly convergent version of Theorem 5.1.

**Remark 5.4** Consider the proof of Theorem 5.1. We deduce from Proposition 5.2(i) that $\boldsymbol{x}_n - \boldsymbol{a}_n \to \boldsymbol{0}$ and, thus, that $\boldsymbol{a}_n \rightharpoonup \boldsymbol{x}$. Moreover, by (5.3), given $i \in I$, the sequence $(a_{i,n})_{n \in \mathbb{N}}$ lies in $\operatorname{dom} \partial\varphi_i \subset \operatorname{dom} \varphi_i$. In particular, if a constraint on $x_i$ is enforced via $\varphi_i = \iota_{C_i}$, then $(a_{i,n})_{n \in \mathbb{N}}$ converges weakly to the $i$th component of a solution $\boldsymbol{x}$ while being feasible in the sense that $C_i \ni a_{i,n} \rightharpoonup x_i$.

## References


[1] F. Acker and M. A. Prestel, Convergence d'un schéma de minimisation alternée, *Ann. Fac. Sci. Toulouse V. Sér. Math.* vol. 2, pp. 1–9, 1980.

[2] S. Alwadani, H. H. Bauschke, and X. Wang, Attouch-Théra duality, generalized cycles, and gap vectors, *SIAM J. Optim.*, vol. 31, pp. 1926–1946, 2021.

[3] A. Argyriou, R. Foygel, and N. Srebro, Sparse prediction with the $k$-support norm, *Proc. Adv. Neural Inform. Process. Syst. Conf.*, vol. 25, pp. 1457–1465, 2012.

[4] H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, Alternating proximal algorithms for weakly coupled convex minimization problems. Applications to dynamical games and PDE's, *J. Convex Anal.*, vol. 15, pp. 485–506, 2008.

[5] J.-B. Baillon, P. L. Combettes, and R. Cominetti, There is no variational characterization of the cycles in the method of periodic projections, *J. Funct. Anal.*, vol. 262, pp. 400–408, 2012.

[6] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 2nd ed. Springer, New York, 2017.

[7] A. Bensoussan, J.-L. Lions, and R. Temam, Sur les méthodes de décomposition, de décentralisation et de coordination, et applications, *Cahier IRIA*, no. 11, pp. 5–189, 1972.

[8] E. Börgens and C. Kanzow, ADMM-Type methods for generalized Nash equilibrium problems in Hilbert spaces, *SIAM J. Optim.*, vol. 31, pp. 377–403, 2021.

[9] A. Borzì and C. Kanzow, Formulation and numerical solution of Nash equilibrium multiobjective elliptic control problems, *SIAM J. Control Optim.*, vol. 51, pp. 718–744, 2013.

[10] R. I. Boţ, S.-M. Grad, and G. Wanka, Maximal monotonicity for the precomposition with a linear operator, *SIAM J. Optim.*, vol. 17, pp. 1239–1252, 2007.

[11] L. M. Briceño-Arias, G. Chierchia, E. Chouzenoux, and J.-C. Pesquet, A random block-coordinate Douglas–Rachford splitting method with low computational complexity for binary logistic regression, *Comput. Optim. Appl.*, vol. 72, pp. 707–726, 2019.

[12] L. M. Briceño-Arias and P. L. Combettes, Convex variational formulation with smooth coupling for multicomponent signal decomposition and recovery, *Numer. Math. Theory Methods Appl.*, vol. 2, pp. 485–508, 2009.

[13] L. M. Briceño-Arias and P. L. Combettes, Monotone operator methods for Nash equilibria in non-potential games, in: *Computational and Analytical Mathematics*, (D. Bailey et al., eds.), pp. 143–159. Springer, New York, 2013.

[14] L. M. Briceño-Arias, P. L. Combettes, J.-C. Pesquet, and N. Pustelnik, Proximal algorithms for multicomponent image recovery problems, *J. Math. Imaging Vision*, vol. 41, pp. 3–22, 2011.

[15] M. N. Bùi and P. L. Combettes, Warped proximal iterations for monotone inclusions, *J. Math. Anal. Appl.*, vol. 491, art. 124315, 21 pp., 2020.

[16] M. N. Bùi and P. L. Combettes, Multivariate monotone inclusions in saddle form, *Math. Oper. Res.*, published online 2021-12-21.

[17] G. Cohen, Nash equilibria: Gradient and decomposition algorithms, *Large Scale Syst.*, vol. 12, pp. 173–184, 1987.

[18] P. L. Combettes, Systems of structured monotone inclusions: Duality, algorithms, and applications, *SIAM J. Optim.*, vol. 23, pp. 2420–2447, 2013.

[19] P. L. Combettes and J. Eckstein, Asynchronous block-iterative primal-dual decomposition methods for monotone inclusions, *Math. Program.*, vol. B168, pp. 645–672, 2018.

[20] P. L. Combettes and J.-C. Pesquet, Fixed point strategies in data science, *IEEE Trans. Signal Process.*, vol. 69, pp. 3878–3905, 2021.

[21] R. Cominetti, F. Facchinei, and J. B. Lasserre, *Modern Optimization Modelling Techniques*. Birkhäuser, Basel, 2012.

[22] J. Darbon and T. Meng, On decomposition models in imaging sciences and multi-time Hamilton–Jacobi partial differential equations, *SIAM J. Imaging Sci.*, vol. 13, pp. 971–1014, 2020.

[23] I. Ekeland and R. Temam, *Analyse Convexe et Problèmes Variationnels.* Dunod, Paris, 1974. English translation: *Convex Analysis and Variational Problems.* SIAM, Philadelphia, PA, 1999.

[24] F. Facchinei, A. Fischer, and V. Piccialli, On generalized Nash games and variational inequalities, *Oper. Res. Lett.*, vol. 35, pp. 159–164, 2007.

[25] P. Gautam, D. R. Sahu, A. Dixit, and T. Som, Forward-backward-half forward dynamical systems for monotone inclusion problems with application to v-GNE, *J. Optim. Theory Appl.*, vol. 190, pp. 491–523, 2021.

[26] A. Habbal and M. Kallel, Neumann–Dirichlet Nash strategies for the solution of elliptic Cauchy problems, *SIAM J. Control Optim.*, vol. 51, pp. 4066–4083, 2013.

[27] Y. He and R. D. C. Monteiro, Accelerating block-decomposition first-order methods for solving composite saddle-point and two-player Nash equilibrium problems, *SIAM J. Optim.*, vol. 25, pp. 2182–2211, 2015.

[28] A. von Heusinger and C. Kanzow, Relaxation methods for generalized Nash equilibrium problems with inexact line search, *J. Optim. Theory Appl.*, vol. 143, pp. 159–183, 2009.

[29] C. Kanzow, V. Karl, D. Steck, and D. Wachsmuth, The multiplier-penalty method for generalized Nash equilibrium problems in Banach spaces, *SIAM J. Optim.*, vol. 29, pp. 767–793, 2019.

[30] J. F. Nash, Jr., Equilibrium points in $n$-person games, *Proc. Nat. Acad. Sci. USA*, vol. 36, pp. 48–49, 1950.

[31] J. Nash, Non-cooperative games, *Ann. Math.*, vol. 54, pp. 286–295, 1951.

[32] A. Nemirovski, Prox-method with rate of convergence $O(1/t)$ for variational inequalities with Lipschitz continuous monotone operators and smooth convex-concave saddle point problems, *SIAM J. Optim.*, vol. 15, pp. 229–251, 2004.

[33] R. T. Rockafellar, Monotone operators associated with saddle-functions and minimax problems, in: *Nonlinear Functional Analysis, Part 1*, (F. E. Browder, ed.), pp. 241–250. AMS, Providence, RI, 1970.

[34] R. T. Rockafellar, Saddle-points and convex analysis, in: *Differential Games and Related Topics*, (H. W. Kuhn and G. P. Szegö, eds.), pp. 109–127. North-Holland, Amsterdam, 1971.

[35] R. T. Rockafellar, Monotone relations and network equilibrium, in: *Variational Inequalities and Network Equilibrium Problems*, (F. Giannessi and A. Maugeri, eds.), pp. 271–288. Plenum Press, New York, 1995.

[36] R. T. Rockafellar, Applications of convex variational analysis to Nash equilibrium, *Proc. 7th Int. Conf. on Nonlinear Analysis and Convex Analysis (Busan, South Korea)*, vol. II, pp. 173–183, 2011.

[37] S. Sorin and C. Wan, Finite composite games: Equilibria and dynamics, *J. Dyn. Games*, vol. 3, pp. 101–120, 2016.

[38] K. K. Thekumparampil, P. Jain, P. Netrapalli, and S. Oh, Efficient algorithms for smooth minimax optimization, *Proc. Adv. Neural Inform. Process. Syst. Conf.*, vol. 32, 2019.

[39] T. Yoon and E. K. Ryu, Accelerated algorithms for smooth convex-concave minimax problems with $\mathcal{O}(1/k^2)$ rate on squared gradient norm, *Proc. 38th Int. Conf. Machine Learn.*, pp. 12098–12109, 2021.