Error Bounds for Strongly Monotone and Lipschitz Continuous Variational Inequalities

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December 19, 2016

Abstract Our aim is to establish lower and upper error bounds for strongly monotone variational inequalities satisfying the Lipschitz continuity. In univariate case, the latter is not needed for getting an upper error bound and a lower error bound is proved by solely using the Lipschitz continuity.

Keywords Variational inequality, strong monotonicity, Lipschitz continuity, calmness, lower error bound, upper error bound, univariate case

Mathematics Subject Classification (2010) 47H05, 47J20.

1 Introduction

Variational inequality (VI for brevity) provides a broad unifying setting for the study of many problems in the field of optimization and equilibrium theory. It also serves as the main computational framework for the solutions of many problems in mathematical sciences.

An important topic in the study of VIs concerns error bounds for estimating the closeness of an arbitrary test vector to the solution set of the given problem. Error bounds have been known to play important roles in proving the convergence and complexity of algorithms. Such error bounds also serve as termination criteria for iterative algorithms and can be used to estimate the amount of error allowable in an inexact computation of the iterates. The reader is referred to [2, Chapter 6] for an introduction to the theory of error bounds for VIs, and to [2, Section 12.6] for the applications of theory error bounds in rate of convergence analysis. Several subsequent results can be found in [1, 3, 5, 8, 10, 11, 12] and the references therein.

In view of the natural residue of the projection equation, Pang [9] obtained an upper error bound, called the global projection-type error bound, for strongly monotone and Lipschitz continuous VIs. This result was extended to ξ-monotone ($\xi > 1$) and Lipschitz continuous VIs in [2, Theorem 2.3.3], and to strongly pseudomonotone and Lipschitz continuous ones in [6, Theorem 4.2].

Motivated by the cited work of Pang, in this paper, we present and prove a sharper lower error bound as well as an upper one for strongly monotone VIs satisfying the Lipschitz continuity. Our lower error bound is new and the upper error bound is sharper than one in [9]. Moreover, in one-dimensional setting,

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we establish an upper error bound for merely strongly monotone VIs and a lower error bound for merely Lipschitz continuous ones.

2 Preliminaries

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. Let K be a nonempty closed convex subset of $\mathbb{R}^n$. For each $u \in \mathbb{R}^n$, there exists a unique point (see [7, Lemma 2.1, p.8]), denoted by $P_K(u)$, such that $\|u - P_K(u)\| \leq \|u - v\|$ for all $v \in K$. Note that $P_K(u) = u$ for every $u \in K$.

Let $F : K \to \mathbb{R}^n$ be a mapping. The variational inequality problem VI(K, F) defined by K and F is to find $u^* \in K$ such that

$$\langle F(u^*), u - u^* \rangle \geq 0, \quad \forall u \in K. \tag{1}$$

Observe that if K is the positive orthant cone, i.e.,

$$K = \mathbb{R}^n_+ := \{u \in \mathbb{R}^n : u_i \geq 0 \ (i = \overline{1, n})\},$$

then u^* is a solution of VI(K, F) if and only if

$$\begin{cases} u^* \in \mathbb{R}^n_+, \ F(u^*) \in \mathbb{R}^n_+, \\ \langle u^*, F(u^*) \rangle = 0. \end{cases}$$

We recall that $F : K \to \mathbb{R}^n$ is said to be *strongly monotone* if there exists $\gamma > 0$ such that

$$\langle F(u) - F(v), u - v \rangle \geq \gamma \|u - v\|^2, \quad \forall u, v \in K, \tag{2}$$

co-coercive if there exists $\beta > 0$ such that

$$\langle F(u) - F(v), u - v \rangle \geq \beta \|F(u) - F(v)\|^2, \quad \forall u, v \in K, \tag{3}$$

Lipschitz continuous if there exists $L > 0$ such that

$$\|F(u) - F(v)\| \leq L \|u - v\|, \quad \forall u, v \in K, \tag{4}$$

and *calm* at $u^* \in K$ if there exists $\kappa > 0$ such that

$$\|F(u) - F(u^*)\| \leq \kappa \|u - u^*\|, \quad \forall u \in K. \tag{5}$$

Remark 2.1

a) It follows from (2), (4) and Cauchy–Schwarz inequality that if F is strongly monotone with modulus γ and Lipschitz continuous with constant L, then F is co-coercive with modulus $\beta = \gamma/L^2$.

b) It follows from (2), (5) and Cauchy–Schwarz inequality that if F is strongly monotone with modulus γ, Lipschitz continuous with constant L, and K has more than one element, then $\gamma \leq L$.

c) Calmness is a "one point" version of the "two points" (Lipschitz continuous) property of mappings. Moreover, if F is Lipschitz continuous on K with constant L, then F is calm at every point $u^* \in K$ with calmness modulus $\kappa = L$.

It is well known that if F is strongly monotone with modulus γ and Lipschitz continuous with constant L, then $\mathrm{VI}(K,F)$ has a unique solution u^* [4, Corollary 3.2] and satisfies the upper error bound [9, Theorem 3.1]

$$\|u-u^*\| \le \left(\frac{L+1}{\gamma}\right)\|u-P_K(u-F(u))\|, \quad \forall u\in K. \tag{6}$$

We observe from the proof of [9, Theorem 3.1] that the author there only needs the calmness of F at u^*, not the Lipschitz continuity. The upper error bound (6) is violated if we relax the strong monotonicity of F by the co-coercivity property (see Remark 3.1(c)).

3 Main results

Theorem 3.1 *Let $K\subset\mathbb{R}^n$ be a nonempty closed convex subset with more than one element. Assume that $F: K\to\mathbb{R}^n$ is strongly monotone with modulus γ and Lipschitz continuous with constant L. Then,* $\mathrm{VI}(K,F)$ *has a unique solution u^* and, for every $u\in K$,*

$$\|u-u^*\| \le \left(\frac{L+1}{2\gamma}+\sqrt{\left(\frac{L+1}{2\gamma}\right)^2-\frac{1}{\gamma}}\right)\|u-P_K(u-F(u))\|, \tag{7}$$

$$\|u-u^*\| \ge \left(\frac{L+1}{2\gamma}-\sqrt{\left(\frac{L+1}{2\gamma}\right)^2-\frac{1}{\gamma}}\right)\|u-P_K(u-F(u))\|. \tag{8}$$

Proof. It follows from [4, Corollary 3.2] that $\mathrm{VI}(K,F)$ has a unique solution u^*. Let u be an arbitrary vector in K and put $v:=u-P_K(u-F(u))$. Then, $u-v=P_K(u-F(u))\in K$. In turn, we deduce from [7, Theorem 2.3] that

$$\langle (u-v)-(u-F(u)),(u-v)-u^*\rangle \le 0$$

or, equivalently,

$$\langle F(u)-v, u-v-u^*\rangle \le 0.$$

Moreover, since u^* is the solution of $\mathrm{VI}(K,F)$, one gets

$$-\langle F(u^*), u-v-u^*\rangle \le 0.$$

Adding these two inequalities and using Cauchy–Schwarz inequality, the strong monotonicity, and the Lipschitz continuity of F, we obtain

$$\begin{aligned}
0 &\ge \langle F(u)-v,u-v-u^*\rangle - \langle F(u^*),u-v-u^*\rangle\\
&= \langle F(u)-F(u^*),u-u^*\rangle + \langle F(u^*)-F(u),v\rangle - \langle v,u-u^*\rangle + \|v\|^2\\
&\ge \gamma\|u-u^*\|^2 - \|F(u)-F(u^*)\|\|v\| - \|u-u^*\|\|v\| + \|v\|^2\\
&\ge \gamma\|u-u^*\|^2 - L\|u-u^*\|\|v\| - \|u-u^*\|\|v\| + \|v\|^2\\
&= \gamma\left(\|u-u^*\|-\frac{L+1}{2\gamma}\|v\|\right)^2 - \left(\gamma\left(\frac{L+1}{2\gamma}\right)^2-1\right)\|v\|^2.
\end{aligned}$$

Hence,

$$\left(\|u-u^*\|-\frac{L+1}{2\gamma}\|v\|\right)^2 \le \left(\left(\frac{L+1}{2\gamma}\right)^2-\frac{1}{\gamma}\right)\|v\|^2. \tag{9}$$

In addition, because $\gamma \le L$ due to Remark 2.1(b), we see that

$$\begin{aligned}\left(\frac{L+1}{2\gamma}\right)^2 - \frac{1}{\gamma} &\ge \left(\frac{\gamma+1}{2\gamma}\right)^2 - \frac{1}{\gamma} \\ &= \frac{(\gamma-1)^2}{4\gamma^2} \ge 0.\end{aligned}$$

Combining this with (9), we obtain (7) and (8). □

Remark 3.1

a) A closer look of the proof of Theorem 3.1 reveals that the Lipschitz continuity of F can be replaced by the calmness of F at u^*, which is a more relaxed condition than Lipschitz continuity. However, the calmness condition has a drawback is that it has to be checked at the solution. The real issue is that we cannot find the exact solution and that is why error bounds are so important since they allow us to estimate the quality of an approximate solution without knowing the actual solution.

b) Observe that

$$\left(\frac{L+1}{2\gamma} + \sqrt{\left(\frac{L+1}{2\gamma}\right)^2 - \frac{1}{\gamma}}\right) - \frac{L+1}{\gamma} = \sqrt{\left(\frac{L+1}{2\gamma}\right)^2 - \frac{1}{\gamma}} - \frac{L+1}{2\gamma} < 0.$$

Hence, the upper error bound (7) is sharper than one in (6).

c) The strong monotonicity of F cannot be omitted in Theorem 3.1. Indeed, consider the VI(K, F) with $K = [0, 1] \subset \mathbb{R}$ and $F(u) = u^2$. Then, $u^* = 0$ is the unique solution of VI(K, F), F is Lipschitz continuous with constant $L = 2$ and co-coercive with modulus $\beta = 1/2$. Indeed, for every $u, v \in K$, we have

$$\begin{aligned}\|F(u) - F(v)\| &= |u^2 - v^2| \\ &= |u-v||u+v| \\ &\le 2|u-v| \\ &= 2\|u-v\|\end{aligned}$$

and

$$\begin{aligned}\langle F(u) - F(v), u - v\rangle &= (u^2 - v^2)(u-v) \\ &= (u-v)^2(u+v) \\ &\ge (1/2)(u-v)^2(u+v)^2 \\ &= (1/2)\|F(u) - F(v)\|^2.\end{aligned}$$

However, it is straightforward to check that F is not strongly monotone. Now for every $u \in K$, note that

$$\|u - u^*\| = |u|, \quad \|u - P_K(u - F(u))\| = |u - (u - u^2)| = u^2.$$

Consequently, since

$$\lim_{u \downarrow 0} \frac{\|u - u^*\|}{\|u - P_K(u - F(u))\|} = \lim_{u \downarrow 0} \frac{|u|}{u^2} = +\infty,$$

one cannot find any $C > 0$ such that

$$\|u - u^*\| \le C\|u - P_K(u - F(u))\|, \quad \forall u \in K. \tag{10}$$

d) The Lipschitz continuity of F cannot be omitted in Theorem 3.1. The following counterexample is available in [2, Example 6.3.4]. Consider the problem VI(K,F) with

$$K = \{(u_1,u_2) \in \mathbb{R}^2 : u_1 \geq 0, u_2 \geq 0\}$$

and

$$F(u) = (u_1 - u_2^2, u_2 + u_2^3), \quad u = (u_1,u_2) \in K.$$

Then, $u^* = (0,0)$ is the unique solution of VI(K,F). Indeed, $u^* = (u_1^*, u_2^*)$ is a solution of VI(K,F) if and only if

$$\begin{cases} u_1^* \geq 0, u_2^* \geq 0, \\ u_1^* - (u_2^*)^2 \geq 0, u_2^* + (u_2^*)^3 \geq 0, \\ u_1^*[u_1^* - (u_2^*)^2] + u_2^*[u_2^* + (u_2^*)^3] = 0. \end{cases}$$

This system is equivalent to

$$\begin{cases} u_1^* \geq 0, u_2^* \geq 0, \\ u_1^* - (u_2^*)^2 \geq 0, u_2^* + (u_2^*)^3 \geq 0, \\ \left[\dfrac{u_1^*}{2} - (u_2^*)^2\right]^2 + \dfrac{3}{4}(u_1^*)^2 + (u_2^*)^2 = 0. \end{cases}$$

Solving this system yields $u_1^* = u_2^* = 0$. Moreover, F is strongly monotone with modulus $\gamma = 1/2$. Indeed, let $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in K, we have

$$\begin{aligned} \langle F(u) - F(v), u - v\rangle &= \langle (u_1 - u_2^2 - v_1 + v_2^2, u_2 + u_2^3 - v_2 - v_2^3), (u_1 - v_1, u_2 - v_2)\rangle \\ &= (u_1 - v_1)^2 - (u_2^2 - v_2^2)(u_1 - v_1) + (u_2 - v_2)^2 + (u_2^3 - v_2^3)(u_2 - v_2) \\ &\geq (u_1 - v_1)^2 - (u_2^2 - v_2^2)(u_1 - v_1) + (u_2 - v_2)^2 + (1/2)(u_2^2 - v_2^2)^2 \\ &\geq (1/2)[(u_1 - v_1)^2 + (u_2 - v_2)^2] + (1/2)[(u_1 - v_1) - (u_2^2 - v_2^2)]^2 \\ &\geq (1/2)[(u_1 - v_1)^2 + (u_2 - v_2)^2] \\ &= (1/2)\|u - v\|^2. \end{aligned}$$

Now let $\{u^k\} \subset K$ given by $u^k = (k^2, k)$ $(k \in \mathbb{N})$. Then, $\|u^k - u^*\| = \sqrt{k^4 + k^2}$ and

$$\begin{aligned} \|u^k - P_K(u^k - F(u^k))\| &= \|(k^2,k) - P_K(k^2, -k^3)\| \\ &= \|(k^2,k) - (k^2,0)\| = k. \end{aligned}$$

It follows that

$$\lim_{k\to+\infty} \frac{\|u^k - u^*\|}{\|u^k - P_K(u^k - F(u^k))\|} = \lim_{k\to+\infty} \frac{\sqrt{k^4 + k^2}}{k} = +\infty.$$

Hence, one cannot find any $C > 0$ satisfying (10).

When $n = 1$, a sharp upper error bound can be set up without the Lipschitz continuity.

Theorem 3.2 *Let $K \subset \mathbb{R}$ be a closed convex subset of $\mathbb{R}^n$ with more than one element. Assume that $F : K \to \mathbb{R}$ is continuous and strongly monotone with modulus γ. Then,* VI(K,F) *has a unique solution u^* and satisfies the upper error bound*

$$|u - u^*| \leq \max\left\{1, \frac{1}{\gamma}\right\} |u - P_K(u - F(u))|, \quad \forall u \in K. \tag{11}$$

Proof. It follows from [4, Corollary 3.2] that VI(K, F) has a unique solution u^*. Since F is strongly monotone with modulus γ, we have

$$(F(u) - F(u^*))(u - u^*) \geq \gamma |u - u^*|^2, \quad \forall u \in K. \tag{12}$$

We consider four possible cases of the set K.

Case 1. $K = \mathbb{R}$.

In this case, we have $F(u^*) = 0$. It follows from (12) that, for every $u \in K$,

$$\begin{aligned} |u - u^*| \leq \frac{1}{\gamma}|F(u) - F(u^*)| &= \frac{1}{\gamma}|F(u)| \\ &= \frac{1}{\gamma}|u - (u - F(u))| \\ &= \frac{1}{\gamma}|u - P_K(u - F(u))|. \end{aligned}$$

Hence, (11) is satisfied.

Case 2. $K = [\alpha, +\infty[$ *for some* $\alpha \in \mathbb{R}$.

Let u be any point in K. It is clear that

$$|u - P_K(u - F(u))| = \begin{cases} |F(u)| & \text{if} \quad u - F(u) \geq \alpha, \\ |u - \alpha| & \text{if} \quad u - F(u) < \alpha. \end{cases}$$

- If $u^* = \alpha$, then $F(u^*) \geq 0$. If $u - F(u) < \alpha$, then

  $$|u - P_K(u - F(u))| = |u - \alpha| = |u - u^*|.$$

  If $u - F(u) \geq \alpha$, then from (12) we have

  $$\begin{aligned} |u - P_K(u - F(u))| = |F(u)| &\geq F(u) - F(u^*) \\ &= |F(u) - F(u^*)| \\ &\geq \gamma |u - u^*|. \end{aligned}$$

  Hence, (11) is satisfied.

- If $u^* > \alpha$, then $F(u^*) = 0$. If $u - F(u) < \alpha$, then

  $$F(u) > u - \alpha \geq 0 = F(u^*),$$

  and we deduce from (12) that $u > u^*$. Thus,

  $$|u - P_K(u - F(u))| = |u - \alpha| > |u - u^*|.$$

  If $u - F(u) \geq \alpha$, then from (12) we have

  $$|u - P_K(u - F(u))| = |F(u)| = |F(u) - F(u^*)| \geq \gamma |u - u^*|.$$

  Hence, (11) is satisfied.

Case 3. $K =]-\infty, \beta]$ *for some* $\beta \in \mathbb{R}$.

Let u be any point in K. It is straightforward to check that

$$|u - P_K(u - F(u))| = \begin{cases} |F(u)| & \text{if} \quad u - F(u) \leq \beta, \\ |u - \beta| & \text{if} \quad u - F(u) > \beta. \end{cases}$$

- If $u^* = \beta$, then $F(u^*) \leq 0$. If $u - F(u) > \beta$, then

$$|u - P_K(u - F(u))| = |u - \beta| = |u - u^*|.$$

If $u - F(u) \leq \beta$, then from (12) we have

$$\begin{aligned} |u - P_K(u - F(u))| = |F(u)| &\geq -F(u) + F(u^*) \\ &= |F(u) - F(u^*)| \\ &\geq \gamma|u - u^*|. \end{aligned}$$

Hence, (11) is satisfied.

- If $u^* < \beta$, then $F(u^*) = 0$. If $u - F(u) > \beta$, then

$$F(u) < u - \beta \leq 0 = F(u^*),$$

and from (12) we have $u < u^*$. Thus,

$$|u - P_K(u - F(u))| = |u - \beta| > |u - u^*|.$$

If $u - F(u) \leq \beta$, then from (12) we have

$$|u - P_K(u - F(u))| = |F(u)| = |F(u) - F(u^*)| \geq \gamma|u - u^*|.$$

Hence, (11) is satisfied.

Case 4. *$K = [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}, \alpha < \beta$.*

Let u be any point in K. We have

$$|u - P_K(u - F(u))| = \begin{cases} |F(u)| & \text{if} \quad \alpha \leq u - F(u) \leq \beta, \\ |u - \beta| & \text{if} \quad u - F(u) > \beta, \\ |u - \alpha| & \text{if} \quad u - F(u) < \alpha. \end{cases}$$

- If $u^* = \alpha$, then $F(u^*) \geq 0$. From (12) we have

$$F(u) \geq F(u^*) \geq 0,$$

and so $u - F(u) \leq \beta$. If $u - F(u) < \alpha$, then

$$|u - P_K(u - F(u))| = |u - \alpha| = |u - u^*|.$$

If $u - F(u) \geq \alpha$, then from (12) we have

$$\begin{aligned} |u - P_K(u - F(u))| = |F(u)| &\geq F(u) - F(u^*) \\ &= |F(u) - F(u^*)| \\ &\geq \gamma|u - u^*|. \end{aligned}$$

Hence, (11) is satisfied.

- If $u^* = \beta$, then $F(u^*) \leq 0$. From (12) we have

$$F(u) \leq F(u^*) \leq 0,$$

and so $u - F(u) \geq \alpha$. If $u - F(u) > \beta$, then

$$|u - P_K(u - F(u))| = |u - \beta| = |u - u^*|.$$

If $u - F(u) \leq \beta$, then from (12) we have

$$\begin{aligned} |u - P_K(u - F(u))| = |F(u)| &\geq -F(u) + F(u^*) \\ &= |F(u) - F(u^*)| \\ &\geq \gamma|u - u^*|. \end{aligned}$$

Hence, (11) is satisfied.

- If $\alpha < u^* < \beta$, then $F(u^*) = 0$. If $u - F(u) < \alpha$, then

$$F(u) > u - \alpha \geq 0 = F(u^*),$$

and from (12) we have $u > u^*$. Thus,

$$|u - P_K(u - F(u))| = |u - \alpha| > |u - u^*|.$$

If $u - F(u) > \beta$, then

$$F(u) < u - \beta \leq 0 = F(u^*),$$

and from (12) we have $u < u^*$. Thus,

$$|u - P_K(u - F(u))| = |u - \beta| > |u - u^*|.$$

If $\alpha \leq u - F(u) \leq \beta$, then from (12) we obtain

$$|u - P_K(u - F(u))| = |F(u)| = |F(u) - F(u^*)| \geq \gamma|u - u^*|.$$

Hence, (11) is satisfied.

□

Remark 3.2 It is interesting to know whether or not we could establish a lower error bound for one-dimensional and strongly monotone VIs. Unfortunately, the answer is negative. To see this, consider the VI(K, F) with $K = \mathbb{R}$ and $F(u) = u + e^u$. Then VI(K, F) has a unique solution u^*, F is continuous and strongly monotone with modulus $\gamma = 1$, but F is not Lipschitz continuous. Now since

$$\lim_{u \to +\infty} \frac{|u - u^*|}{|u - P_K(u - F(u)|} = \lim_{u \to +\infty} \frac{|u - u^*|}{|u + e^u|} = 0,$$

one cannot find any $C > 0$ such that

$$|u - u^*| \geq C|u - P_K(u - F(u))|, \quad \forall u \in K.$$

A small lower error bound is often used as a necessary condition for checking the accuracy of approximate solutions. If a lower error bound is unacceptably large, then the actual approximation errors are even larger, and hence, we reject the hypothesis that an approximate solution is accurate. Replacing the strong monotonicity of the mapping F in Theorem 3.2 by the Lipschitz continuity, we obtain a lower error bound for VIs having a unique solution.

Theorem 3.3 *Let $K \subset \mathbb{R}$ be a nonempty closed convex subset and $F\colon K \to \mathbb{R}$ a Lipschitz continuous mapping with constant L. Suppose that $\mathrm{VI}(K,F)$ has a unique solution, say u^*. Then, for every $u \in K$, we have*

$$\min\left\{1, \frac{1}{L}\right\} |u - P_K(u - F(u))| \leq |u - u^*|. \tag{13}$$

Proof. We consider four conceivable cases of the set K.

(a) $K = \mathbb{R}$: Since $(\forall v \in K)\ F(u^*)\,(v - u^*) \geq 0$, we obtain $F(u^*) = 0$. Combining this and the Lipschitz continuity of F yields

$$\begin{aligned} |u - P_K(u - F(u))| &= |F(u)| = |F(u) - F(u^*)| \\ &\leq L\,|u - u^*|. \end{aligned}$$

Hence, (13) holds.

(b) $K = [\alpha, +\infty[$ for some $\alpha \in \mathbb{R}$: Fix $u \in K$. Then it is straightforward to check that

$$|u - P_K(u - F(u))| = \begin{cases} |F(u)|, & \text{if } u - F(u) \geq \alpha; \\ |u - \alpha|, & \text{otherwise.} \end{cases} \tag{14}$$

Since $(\forall v \in K)\ F(u^*)\,(v - u^*) \geq 0$, we obtain $F(u^*) \geq 0$. We consider two subcases.

(b.1) $F(u^*) = 0$: If $u - F(u) \geq \alpha$, then it follows from (14) and the Lipschitz continuity of F that

$$\begin{aligned} |u - P_K(u - F(u))| &= |F(u)| = |F(u) - F(u^*)| \\ &\leq L\,|u - u^*|; \end{aligned}$$

otherwise, since $F(u) > u - \alpha \geq 0$, we derive from (14) and the Lipschitz continuity of F that

$$\begin{aligned} |u - P_K(u - F(u))| &= |u - \alpha| = u - \alpha \\ &< F(u) \\ &= |F(u) - F(u^*)| \\ &\leq L\,|u - u^*|. \end{aligned}$$

Hence, (13) holds.

(b.2) $F(u^*) > 0$: We first observe that, since $(\forall v \in K)\ F(u^*)\,(v - u^*) \geq 0$ and $F(u^*) > 0$, u^* must be α. In addition, we have $F(u) > 0$. Indeed, if $F(u) \leq 0$, then, since F is continuous on K, the intermediate value theorem guarantees the existence of an element $v^* \in\,]u^*, u]$ such that $F(v^*) = 0$. Obviously, v^* is a solution of $\mathrm{VI}(K,F)$ and $v^* \neq u^*$, which contradicts the assumption that u^* is the unique solution of $\mathrm{VI}(K,F)$. Thus, if $u - F(u) \geq \alpha$, then it follows from (14) that

$$|u - P_K(u - F(u))| = |F(u)| = F(u) \leq u - \alpha = |u - u^*|;$$

otherwise, by (14), we get

$$|u - P_K(u - F(u))| = |u - \alpha| = |u - u^*|.$$

Hence, (13) holds.

(c) $K = \,]-\infty, \beta]$ for some $\beta \in \mathbb{R}$: Fix $u \in K$. Then it is straightforward to check that

$$|u - P_K(u - F(u))| = \begin{cases} |F(u)|, & \text{if } u - F(u) \leq \beta; \\ |u - \beta|, & \text{otherwise.} \end{cases} \tag{15}$$

Since $(\forall v \in K)\ F(u^*)\,(v - u^*) \geq 0$, we must have $F(u^*) \leq 0$. We consider two subcases.

(c.1) $F(u^*) = 0$: If $u - F(u) \leq \beta$, then it follows from (15) and the Lipschitz continuity of F that

$$\begin{aligned}|u - P_K(u - F(u))| &= |F(u)| = |F(u) - F(u^*)| \\ &\leq L\,|u - u^*|\,;\end{aligned}$$

otherwise, since $-F(u) > \beta - u \geq 0$, we derive from (15) and the Lipschitz continuity of F that

$$\begin{aligned}|u - P_K(u - F(u))| = |u - \beta| &= \beta - u \\ &< -F(u) \\ &= |F(u^*) - F(u)| \\ &\leq L\,|u - u^*|\,.\end{aligned}$$

Hence, (13) holds.

(c.2) $F(u^*) < 0$: We first observe that, since $(\forall v \in K)\,, F(u^*)\,(v - u^*) \geq 0$ and $F(u^*) < 0$, u^* must be β. In addition, $F(u)$ must be strictly negative. Indeed, if $F(u) \geq 0$, then, since F is continuous on K, the intermediate value theorem guarantees the existence of an element $v^* \in [u, u^*[$ such that $F(v^*) = 0$. Obviously, v^* is a solution of the problem $\mathrm{VI}(K, F)$ and $v^* \neq u^*$, which contradicts the assumption that u^* is the unique solution of $\mathrm{VI}(K, F)$. Therefore, if $u - F(u) \leq \beta$, then it follows from (15) that

$$|u - P_K(u - F(u))| = |F(u)| = -F(u) \leq \beta - u = |u - u^*|\,;$$

otherwise, by (15), we obtain

$$|u - P_K(u - F(u))| = |u - \beta| = |u - u^*|\,.$$

Hence, (13) holds.

(d) $K = [\alpha, \beta]$ for some α, $\beta \in \mathbb{R}$ with $\alpha < \beta$: Fix $u \in K$. Then it is straightforward to check that

$$|u - P_K(u - F(u))| = \begin{cases} |F(u)|\,, & \text{if } u - F(u) \in [\alpha, \beta]\,; \\ |u - \alpha|\,, & \text{if } u - F(u) < \alpha; \\ |u - \beta|\,, & \text{otherwise.} \end{cases} \tag{16}$$

We consider three subcases.

(d.1) $u^* = \alpha$: Since $(\forall v \in K)\ F(u^*)\,(v - u^*) \geq 0$, we obtain $F(u^*) \geq 0$. We first observe that $F(\beta)$ must be strictly positive. Indeed, if $F(\beta) \leq 0$, then, since $(\forall v \in K)\ v - \beta \leq 0$, we have $(\forall v \in K)\ F(\beta)\,(v - \beta) \geq 0$; hence, β is a solution of $\mathrm{VI}(K, F)$, which violates our hypothesis. Moreover, we must have $F(u) \geq 0$ since, if $F(u) < 0$, then, by the intermediate value theorem, there exists an element $v^* \in]u, \beta[$ such that $F(v^*) = 0$, and, in turn, v^* is a solution of $\mathrm{VI}(K, F)$, which contradicts our assumption. Therefore, $u - F(u) \leq \beta$. If $\alpha \leq u - F(u) \leq \beta$, then it follows from (16) that

$$\begin{aligned}|u - P_K(u - F(u))| = |F(u)| &= F(u) \\ &\leq u - \alpha \\ &= |u - u^*|\,;\end{aligned}$$

otherwise, there is nothing to prove. Hence, (13) holds.

(d.2) $u^* = \beta$: A similar fashion to the case (d.1).

(d.3) $u^* \in]\alpha, \beta[$: Since $(\forall v \in K)\ F(u^*)(v - u^*) \geq 0$ and $u^* \in]\alpha, \beta[$, we obtain $F(u^*) = 0$. If $\alpha \leq u - F(u) \leq \beta$, then it follows from (16) and the Lipschitz continuity of F that

$$\begin{aligned} |u - P_K(u - F(u))| = |F(u)| &= |F(u) - F(u^*)| \\ &\leq L\,|u - u^*|\,; \end{aligned}$$

if $u - F(u) < \alpha$, then it follows from (16) and the Lipschitz continuity of F that

$$\begin{aligned} |u - P_K(u - F(u))| = |u - \alpha| &= u - \alpha \\ &< F(u) \\ &= F(u) - F(u^*) \\ &\leq L\,|u - u^*|\,; \end{aligned}$$

otherwise, it follows from (16) and the Lipschitz continuity of F that

$$\begin{aligned} |u - P_K(u - F(u))| = |u - \beta| &= \beta - u \\ &\leq -F(u) \\ &= F(u^*) - F(u) \\ &\leq L\,|u - u^*|\,. \end{aligned}$$

Hence, (13) holds. □

4 Conclusions

We have established the sharp error bounds for strongly monotone and Lipschitz continuous VIs in 1-dimensional case and higher dimensional setting. The result and the proof of Theorem 3.1 are still valid if $\mathbb{R}^n$ is replaced by a real Hilbert space.

Acknowledgements

The first author is funded by Center for Mathematical Modeling, Universidad de Chile. The authors would like to thank Prof. Nguyen Dong Yen for valuable remarks and suggestions.

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