A Decomposition Method for Solving Multicommodity Network Equilibria

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Abstract. We consider the numerical aspect of the multicommodity network equilibrium problem proposed by Rockafellar in 1995. Our method relies on the flexible monotone operator splitting framework recently proposed by Combettes and Eckstein.

Keywords. Network equilibrium, network flow, traffic assignment, splitting algorithm, block-iterative algorithm, monotone operator.

1 Problem formulation

Rockafellar proposed in [21] the important multicommodity network equilibrium model (see (6) in Problem 2) and studied some of its properties. In the present paper, we devise a flexible numerical method for solving this problem based on the asynchronous block-iterative decomposition framework of [8].

The following notion of a network from [20, Section 1A] plays a central role in the formulation of our problem.

Definition 1 A network consists of nonempty finite sets $\mathscr{N}$ and $\mathscr{A}$ — whose elements are called nodes and arcs, respectively — and a mapping $\vartheta\colon \mathscr{A} \to \mathscr{N} \times \mathscr{N} : j \mapsto (\vartheta_1(j), \vartheta_2(j))$ such that, for every $j \in \mathscr{A}$, $\vartheta_1(j) \neq \vartheta_2(j)$. We call $\vartheta_1(j)$ and $\vartheta_2(j)$ the initial node and the terminal node of arc j, respectively. In addition, we set

$$(\forall i \in \mathscr{N}) \quad \begin{cases} \mathscr{A}^+(i) = \big\{ j \in \mathscr{A} \mid \text{node } i \text{ is the initial node of arc } j \big\} \\ \mathscr{A}^-(i) = \big\{ j \in \mathscr{A} \mid \text{node } i \text{ is the terminal node of arc } j \big\}. \end{cases} \tag{1}$$

Recall that, given a Euclidean space $\mathcal{G}$ with scalar product $\langle \cdot \mid \cdot \rangle$, an operator $A\colon \mathcal{G} \to 2^{\mathcal{G}}$ is maximally monotone if

$$\big(\forall (x, x^*) \in \mathcal{G} \times \mathcal{G}\big) \quad (x, x^*) \in \operatorname{gra} A \quad \Leftrightarrow \quad \big[\, \big(\forall (y, y^*) \in \operatorname{gra} A\big) \;\; \langle x - y \mid x^* - y^* \rangle \geqslant 0 \,\big], \tag{2}$$

where $\operatorname{gra} A = \big\{ (x, x^*) \in \mathcal{G} \times \mathcal{G} \mid x^* \in Ax \big\}$ is the graph of A. (The reader is referred to [2] for background and complements on monotone operator theory and convex analysis.) The problem of interest is the following.

Problem 2 Under consideration is a network $(\mathscr{N},\mathscr{A},\vartheta)$, together with a nonempty finite set $\mathscr{C}$ of commodities transiting on the network. Equip $\mathcal{H}=\mathbb{R}^{\mathscr{C}}$ with the scalar product $((\xi_k)_{k\in\mathscr{C}},(\eta_k)_{k\in\mathscr{C}})\mapsto\sum_{k\in\mathscr{C}}\xi_k\eta_k$ and let us introduce the spaces

$$\begin{cases}\boldsymbol{\mathcal{X}}=\big\{\boldsymbol{x}=(x_j)_{j\in\mathscr{A}}\mid(\forall j\in\mathscr{A})\;\;x_j=(\xi_{j,k})_{k\in\mathscr{C}}\in\mathcal{H}\big\}\\ \boldsymbol{\mathcal{V}}=\big\{\boldsymbol{v}^*=(v_i^*)_{i\in\mathscr{N}}\mid(\forall i\in\mathscr{N})\;\;v_i^*=(\nu_{i,k}^*)_{k\in\mathscr{C}}\in\mathcal{H}\big\}.\end{cases}\tag{3}$$

An element $\boldsymbol{x}\in\boldsymbol{\mathcal{X}}$ is called a flow on the network, where $\xi_{j,k}$ is the flux of commodity k on arc j. The divergence of a flow $\boldsymbol{x}\in\boldsymbol{\mathcal{X}}$ at node i is

$$\operatorname{div}_i\boldsymbol{x}=\sum_{j\in\mathscr{A}^+(i)}x_j-\sum_{j\in\mathscr{A}^-(i)}x_j.\tag{4}$$

We refer to an element $\boldsymbol{v}^*\in\boldsymbol{\mathcal{V}}$ as a potential on the network, where $\nu_{i,k}^*$ is the potential of commodity k at node i. Given $\boldsymbol{v}^*\in\boldsymbol{\mathcal{V}}$ and $j\in\mathscr{A}$, the tension (or potential difference) across arc j relative to the potential $\boldsymbol{v}^*$ is

$$\Delta_j\boldsymbol{v}^*=v^*_{\vartheta_2(j)}-v^*_{\vartheta_1(j)}.\tag{5}$$

For every $j\in\mathscr{A}$, the flow-tension relation on arc j is modeled by the sum Q_j+R_j of maximally monotone operators $Q_j\colon\mathcal{H}\to2^{\mathcal{H}}$ and $R_j\colon\mathcal{H}\to2^{\mathcal{H}}$. Further, for every $i\in\mathscr{N}$, the divergence-potential relation at node i is modeled by a maximally monotone operator $S_i\colon\mathcal{H}\to2^{\mathcal{H}}$. The task is to

$$\text{find a flow }\;\overline{\boldsymbol{x}}\in\boldsymbol{\mathcal{X}}\;\text{ and a potential }\;\overline{\boldsymbol{v}}^*\in\boldsymbol{\mathcal{V}}\;\text{ such that }\;\begin{cases}(\forall j\in\mathscr{A})\;\;\Delta_j\overline{\boldsymbol{v}}^*\in Q_j\overline{x}_j+R_j\overline{x}_j\\(\forall i\in\mathscr{N})\;\;\operatorname{div}_i\overline{\boldsymbol{x}}\in S_i^{-1}\overline{v}_i^*,\end{cases}\tag{6}$$

under the assumption that (6) has a solution.

Remark 3 The pertinence of Problem 2 is demonstrated in [20, Chapter 8] and [21], where it is shown to capture formulations arising in areas such as traffic assignment, hydraulic networks, and price equilibrium.

2 A block-iterative decomposition method

Notation. Throughout, $\mathcal{G}$ is a Euclidean space. Let $A\colon\mathcal{G}\to2^{\mathcal{G}}$ be maximally monotone and let $x\in\mathcal{G}$. Then, in terms of the variable $p\in\mathcal{G}$, the inclusion $x\in p+Ap$ has a unique solution, which is denoted by J_Ax. The operator $J_A\colon\mathcal{G}\to\mathcal{G}\colon x\mapsto J_Ax$ is called the resolvent of A.

Our algorithm (see (8) in Proposition 4) is derived from [8, Algorithm 12] and it thus inherits the following attractive features from the framework of [8]:

① No additional assumption, such as Lipschitz continuity or cocoercivity, is imposed on the underlying operators.

② Algorithm (8) achieves full splitting in the sense that the operators $(Q_j)_{j\in\mathscr{A}}$, $(R_j)_{j\in\mathscr{A}}$, and $(S_i)_{i\in\mathscr{N}}$ are activated independently via their resolvents.

③ Algorithm (8) is block-iterative, that is, at iteration n, only blocks $(Q_j)_{j\in\mathscr{A}_n}$, $(R_j)_{j\in\mathscr{A}_n}$, and $(S_i)_{i\in\mathscr{N}_n}$ of operators need to be activated, where $\mathscr{A}_n$ and $\mathscr{N}_n$ are nonempty subsets of $\mathscr{A}$ and $\mathscr{N}$, respectively. To guarantee convergence of the iterates, the mild sweeping condition (7) below needs to be fulfilled.

We denote elements in $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{V}}$ by bold letters, e.g., $\boldsymbol{q}_n = (q_{j,n})_{j\in\mathscr{A}}$ and $\boldsymbol{s}_n^* = (s_{i,n}^*)_{i\in\mathscr{N}}$.

Proposition 4 *Consider the setting of Problem 2. Let $T \in \mathbb{N}$, let $(\mathscr{A}_n)_{n\in\mathbb{N}}$ be nonempty subsets of $\mathscr{A}$, and let $(\mathscr{N}_n)_{n\in\mathbb{N}}$ be nonempty subsets of $\mathscr{N}$ such that $\mathscr{A}_0 = \mathscr{A}$, $\mathscr{N}_0 = \mathscr{N}$, and*

$$(\forall n \in \mathbb{N}) \quad \bigcup_{k=n}^{n+T} \mathscr{A}_k = \mathscr{A} \quad \text{and} \quad \bigcup_{k=n}^{n+T} \mathscr{N}_k = \mathscr{N}. \tag{7}$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0,2[$ such that $\inf_{n\in\mathbb{N}} \lambda_n > 0$ and $\sup_{n\in\mathbb{N}} \lambda_n < 2$. Moreover, for every $j \in \mathscr{A}$ and every $i \in \mathscr{N}$, let $(x_{j,0}, x_{j,0}^*, v_{i,0}^*) \in \mathcal{H}^3$ and $(\gamma_j, \mu_j, \sigma_i) \in]0,+\infty[^3$. Iterate

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for every } j \in \mathscr{A}_n\\
\left\lfloor
\begin{array}{l}
l_{j,n}^* = x_{j,n}^* - \Delta_j \boldsymbol{v}_n^*\\
q_{j,n} = J_{\gamma_j Q_j}(x_{j,n} - \gamma_j l_{j,n}^*)\\
q_{j,n}^* = \gamma_j^{-1}(x_{j,n} - q_{j,n}) - l_{j,n}^*\\
r_{j,n} = J_{\mu_j R_j}(x_{j,n} + \mu_j x_{j,n}^*)\\
r_{j,n}^* = x_{j,n}^* + \mu_j^{-1}(x_{j,n} - r_{j,n})
\end{array}
\right.\\
\text{for every } j \in \mathscr{A} \smallsetminus \mathscr{A}_n\\
\lfloor\; q_{j,n} = q_{j,n-1};\; q_{j,n}^* = q_{j,n-1}^*;\; r_{j,n} = r_{j,n-1};\; r_{j,n}^* = r_{j,n-1}^*\\
\text{for every } i \in \mathscr{N}_n\\
\left\lfloor
\begin{array}{l}
l_{i,n} = \operatorname{div}_i \boldsymbol{x}_n\\
s_{i,n} = J_{\sigma_i S_i}(l_{i,n} + \sigma_i v_{i,n}^*)\\
s_{i,n}^* = v_{i,n}^* + \sigma_i^{-1}(l_{i,n} - s_{i,n})\\
t_{i,n} = s_{i,n} - \operatorname{div}_i \boldsymbol{q}_n
\end{array}
\right.\\
\text{for every } i \in \mathscr{N} \smallsetminus \mathscr{N}_n\\
\left\lfloor
\begin{array}{l}
s_{i,n} = s_{i,n-1};\; s_{i,n}^* = s_{i,n-1}^*\\
t_{i,n} = s_{i,n} - \operatorname{div}_i \boldsymbol{q}_n
\end{array}
\right.\\
\text{for every } j \in \mathscr{A}\\
\left\lfloor
\begin{array}{l}
t_{j,n}^* = q_{j,n}^* + r_{j,n}^* - \Delta_j \boldsymbol{s}_n^*\\
u_{j,n} = r_{j,n} - q_{j,n}
\end{array}
\right.\\
\tau_n = \sum_{j\in\mathscr{A}} \left(\|t_{j,n}^*\|^2 + \|u_{j,n}\|^2\right) + \sum_{i\in\mathscr{N}} \|t_{i,n}\|^2\\
\text{if } \tau_n > 0\\
\left\lfloor
\begin{array}{l}
\pi_n = \sum_{j\in\mathscr{A}} \left(\langle x_{j,n} \mid t_{j,n}^*\rangle - \langle q_{j,n} \mid q_{j,n}^*\rangle + \langle u_{j,n} \mid x_{j,n}^*\rangle - \langle r_{j,n} \mid r_{j,n}^*\rangle\right)\\
\qquad + \sum_{i\in\mathscr{N}} \left(\langle t_{i,n} \mid v_{i,n}^*\rangle - \langle s_{i,n} \mid s_{i,n}^*\rangle\right)\\
\theta_n = \lambda_n \max\{\pi_n, 0\}/\tau_n
\end{array}
\right.\\
\text{else}\\
\lfloor\; \theta_n = 0\\
\text{for every } j \in \mathscr{A}\\
\left\lfloor
\begin{array}{l}
x_{j,n+1} = x_{j,n} - \theta_n t_{j,n}^*\\
x_{j,n+1}^* = x_{j,n}^* - \theta_n u_{j,n}
\end{array}
\right.\\
\text{for every } i \in \mathscr{N}\\
\lfloor\; v_{i,n+1}^* = v_{i,n}^* - \theta_n t_{i,n}.
\end{array}
\right.
\end{array}
\tag{8}
$$

Then $((x_{j,n})_{j\in\mathscr{A}}, (v_{i,n}^*)_{i\in\mathscr{N}})_{n\in\mathbb{N}}$ converges to a solution to (6).

Proof. Let us consider the multivariate monotone inclusion problem

$$\text{find } \overline{\boldsymbol{x}} \in \boldsymbol{\mathcal{X}},\ \overline{\boldsymbol{x}}^* \in \boldsymbol{\mathcal{X}},\ \text{and } \overline{\boldsymbol{v}}^* \in \boldsymbol{\mathcal{V}} \text{ such that } \begin{cases} (\forall j \in \mathcal{A})\ \ \Delta_j \overline{\boldsymbol{v}}^* - \overline{x}_j^* \in Q_j \overline{x}_j \ \text{ and } \ \overline{x}_j \in R_j^{-1} \overline{x}_j^* \\ (\forall i \in \mathcal{N})\ \ \operatorname{div}_i \overline{\boldsymbol{x}} \in S_i^{-1} \overline{v}_i^*. \end{cases} \tag{9}$$

Then

$$\begin{aligned} (\forall \overline{\boldsymbol{x}} \in \boldsymbol{\mathcal{X}})(\forall \overline{\boldsymbol{v}}^* \in \boldsymbol{\mathcal{V}}) \quad & (\overline{\boldsymbol{x}}, \overline{\boldsymbol{v}}^*) \ \text{solves (6)} \\ & \Leftrightarrow (\exists\, \overline{\boldsymbol{x}}^* \in \boldsymbol{\mathcal{X}}) \ \begin{cases} (\forall j \in \mathcal{A})\ \ \Delta_j \overline{\boldsymbol{v}}^* \in Q_j \overline{x}_j + \overline{x}_j^* \ \text{ and } \ \overline{x}_j^* \in R_j \overline{x}_j \\ (\forall i \in \mathcal{N})\ \ \operatorname{div}_i \overline{\boldsymbol{x}} \in S_i^{-1} \overline{v}_i^* \end{cases} \\ & \Leftrightarrow (\exists\, \overline{\boldsymbol{x}}^* \in \boldsymbol{\mathcal{X}}) \ (\overline{\boldsymbol{x}}, \overline{\boldsymbol{x}}^*, \overline{\boldsymbol{v}}^*) \ \text{solves (9)}. \end{aligned} \tag{10}$$

Therefore, since (6) has a solution, so does (9). Next, define

$$(\forall i \in \mathcal{N})(\forall j \in \mathcal{A}) \quad \varepsilon_{i,j} = \begin{cases} 1, & \text{if node } i \text{ is the initial node of arc } j; \\ -1, & \text{if node } i \text{ is the terminal node of arc } j; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

It results from (4) and (1) that

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}})(\forall i \in \mathcal{N}) \quad \operatorname{div}_i \boldsymbol{x} = \sum_{j \in \mathcal{A}} \varepsilon_{i,j} x_j, \tag{12}$$

and from (5) that

$$(\forall \boldsymbol{v}^* \in \boldsymbol{\mathcal{V}})(\forall j \in \mathcal{A}) \quad \Delta_j \boldsymbol{v}^* = -\sum_{i \in \mathcal{N}} \varepsilon_{i,j} v_i^*. \tag{13}$$

We now verify that (9) is a special case of [8, Problem 1] with the setting $I = \mathcal{A}$, $K = \mathcal{A} \cup \mathcal{N}$, and for every $j \in I$ and every $k \in K$,

$$\mathcal{H}_j = \mathcal{G}_k = \mathcal{H}, \quad A_j = Q_j, \quad z_j^* = r_k = 0, \quad \text{and} \quad \begin{cases} B_k = \begin{cases} R_k, & \text{if } k \in \mathcal{A}; \\ S_k, & \text{if } k \in \mathcal{N} \end{cases} \\ L_{k,j} = \begin{cases} \mathrm{Id}, & \text{if } k = j; \\ 0, & \text{if } k \in \mathcal{A} \text{ and } k \neq j; \\ \varepsilon_{k,j} \mathrm{Id}, & \text{if } k \in \mathcal{N}. \end{cases} \end{cases} \tag{14}$$

We deduce from (12) that

$$\begin{aligned} (\forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}})(\forall k \in K) \quad \sum_{j \in I} L_{k,j} x_j &= \begin{cases} x_k, & \text{if } k \in \mathcal{A}; \\ \sum_{j \in I} \varepsilon_{k,j} x_j, & \text{if } k \in \mathcal{N} \end{cases} \\ &= \begin{cases} x_k, & \text{if } k \in \mathcal{A}; \\ \operatorname{div}_k \boldsymbol{x}, & \text{if } k \in \mathcal{N}, \end{cases} \end{aligned} \tag{15}$$

and from (13) that

$$(\forall \boldsymbol{x}^* \in \boldsymbol{\mathcal{X}})(\forall \boldsymbol{v}^* \in \boldsymbol{\mathcal{V}})(\forall j \in I) \quad \sum_{k \in \mathcal{A}} L_{k,j}^* x_k^* + \sum_{k \in \mathcal{N}} L_{k,j}^* v_k^* = x_j^* + \sum_{k \in \mathcal{N}} \varepsilon_{k,j} v_k^* = x_j^* - \Delta_j \boldsymbol{v}^*. \tag{16}$$

Hence, in the setting of (14), (9) is an instantiation of [8, Problem 1] and (8) is a realization of [8, Algorithm 12], where $(\forall n \in \mathbb{N})$ $I_n = \mathcal{A}_n$ and $K_n = \mathcal{A}_n \cup \mathcal{N}_n$. Thus, upon letting

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{x}_n = (x_{j,n})_{j \in \mathcal{A}}, \quad \boldsymbol{x}_n^* = (x_{j,n}^*)_{j \in \mathcal{A}}, \quad \text{and} \quad \boldsymbol{v}_n^* = (v_{i,n}^*)_{i \in \mathcal{N}}, \tag{17}$$

we infer from [8, Theorem 13] that $(\boldsymbol{x}_n, \boldsymbol{x}_n^*, \boldsymbol{v}_n^*)_{n \in \mathbb{N}}$ converges to a solution $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{x}}^*, \overline{\boldsymbol{v}}^*)$ to (9). Consequently, (10) asserts that $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{v}}^*)$ solves (6). ☐

Remark 5 Some comments are in order.

(i) One might be tempted to consider (6) as a special case of [8, Problem 1] with the setting $I = \mathscr{A}$, $K = \mathscr{N}$, and for every $j \in I$ and every $k \in K$,

$$\mathcal{H}_j = \mathcal{G}_k = \mathcal{H}, \quad A_j = Q_j + R_j, \quad B_k = S_k, \quad z_j^* = r_k = 0, \quad \text{and} \quad L_{k,j} = \varepsilon_{k,j}\mathrm{Id}, \tag{18}$$

where $(\varepsilon_{i,j})_{i\in\mathscr{N},j\in\mathscr{A}}$ are defined in (11), and then specialize [8, Algorithm 12] to (18). However, this approach necessitates the computation of the resolvents of the operators $(Q_j + R_j)_{j\in\mathscr{A}}$, which cannot be expressed in terms of the resolvents of $(Q_j)_{j\in\mathscr{A}}$ and $(R_j)_{j\in\mathscr{A}}$ in general (see Examples 7 and 8).

(ii) Algorithm (8) of Proposition 4 requires to evaluate the resolvents of the operators $(Q_j)_{j\in\mathscr{A}}$, $(R_j)_{j\in\mathscr{A}}$, and $(S_i)_{i\in\mathscr{N}}$. Illustrations of such calculations are provided in Examples 6–8 and 11–14.

(iii) Alternate algorithms [7, 9, 19] can also be used to solve (9) and, in turn, (6). Nevertheless, there are certain restrictions on the resulting algorithms. For example, the method of [7] must activate all the operators $(Q_j)_{j\in\mathscr{A}}$, $(R_j)_{j\in\mathscr{A}}$, and $(S_i)_{i\in\mathscr{N}}$ at every iteration, while the frameworks of [9, 19] do not allow for deterministic selections of the blocks $(Q_j)_{j\in\mathscr{A}_n}$, $(R_j)_{j\in\mathscr{A}_n}$, and $(S_i)_{i\in\mathscr{N}_n}$. Finally, the algorithm resulted from [9] involves the inversion of a linear operator acting on $\mathbb{R}^{MN}$, where $M = \operatorname{card}\mathscr{A}$ and $N = \operatorname{card}\mathscr{C}$, which may not be favorable in large-scale problems, e.g., [12].

Notation. Before proceeding further, let us recall some basic notion of convex analysis (see [2] for details). Let $\varphi\colon \mathcal{G} \to \,]-\infty,+\infty]$ be proper, lower semicontinuous, and convex. The subdifferential of φ is the maximally monotone operator $\partial\varphi\colon \mathcal{G} \to 2^{\mathcal{G}}\colon x \mapsto \big\{x^* \in \mathcal{G} \mid (\forall y \in \mathcal{G})\ \langle y - x \mid x^*\rangle + \varphi(x) \leqslant \varphi(y)\big\}$. For every $x \in \mathcal{G}$, the unique minimizer of $\varphi + (1/2)\|\cdot - x\|^2$ is denoted by $\operatorname{prox}_\varphi x$. Let C be a nonempty closed convex subset of $\mathcal{G}$. The indicator function of C is the proper lower semicontinuous convex function

$$\iota_C\colon \mathcal{G} \to [0,+\infty] : x \mapsto \begin{cases} 0, & \text{if } x \in C;\\ +\infty, & \text{otherwise,}\end{cases} \tag{19}$$

the normal cone operator of C is $N_C = \partial\iota_C$, and the projector onto C is $\operatorname{proj}_C = \operatorname{prox}_{\iota_C}$.

To motivate the need for monotone operators in the formulation of Problem 2, let us consider the following example.

Example 6 (Gyrator) The current-voltage relation of an ideal gyrator [22] is modeled by the maximally monotone operator

$$Q\colon \mathbb{R}^2 \to \mathbb{R}^2\colon (\xi_1,\xi_2) \mapsto (-\xi_2,\xi_1). \tag{20}$$

We have

$$\big(\forall\gamma \in \,]0,+\infty[\,\big)\big(\forall(\xi_1,\xi_2) \in \mathbb{R}^2\big) \quad J_{\gamma Q}(\xi_1,\xi_2) = \frac{1}{1+\gamma^2}(\xi_1 + \gamma\xi_2, -\gamma\xi_1 + \xi_2). \tag{21}$$

Note that Q is not a subdifferential. Electrical networks with gyrators are considered in [6].

Example 7 (Separable multicommodity flows) Consider the setting of Problem 2 and suppose, in addition, that the following are satisfied:

[a] For every $j \in \mathscr{A}$, $\mathfrak{c}_j \colon \mathbb{R} \to 2^{\mathbb{R}}$ is maximally monotone, C_j is a nonempty closed convex subset of $\mathcal{H}$, Q_j maps $x_j = (\xi_{j,k})_{k\in\mathscr{C}} \in \mathcal{H}$ to the set $\big\{(\xi_j^*)_{k\in\mathscr{C}} \mid \xi_j^* \in \mathfrak{c}_j(\sum_{k\in\mathscr{C}} \xi_{j,k})\big\}$, and $R_j = N_{C_j}$.

[b] For every $i \in \mathscr{N}$, $s_i \in \mathcal{H}$ and $S_i^{-1} \colon \mathcal{H} \to 2^{\mathcal{H}} \colon v_i^* \mapsto \{s_i\}$.

Then (6) reduces to the separable multicommodity flow problem; see, e.g., [3, Section 8.3] and the references listed in [3, Section 8.9]. Take $j \in \mathscr{A}$, $i \in \mathscr{N}$, and $\gamma \in \left]0, +\infty\right[$. We have $J_{\gamma R_j} = \text{proj}_{C_j}$ and $J_{\gamma S_i} = s_i$. To compute $J_{\gamma Q_j}$, define $L \colon \mathcal{H} \to \mathbb{R} \colon (\xi_k)_{k\in\mathscr{C}} \mapsto \sum_{k\in\mathscr{C}} \xi_k$ and set $N = \text{card}\,\mathscr{C}$. Then $L^* \colon \mathbb{R} \to \mathcal{H} \colon \xi \mapsto (\xi)_{k\in\mathscr{C}}$ and, therefore, $L \circ L^* = N\text{Id}$. At the same time, by [a], $Q_j = L^* \circ \mathfrak{c}_j \circ L$. Thus, we derive from [2, Proposition 23.25(iii)] that

$$\big(\forall x_j = (\xi_{j,k})_{k\in\mathscr{C}} \in \mathcal{H}\big) \quad J_{\gamma Q_j} x_j = x_j + \frac{1}{N}\big(J_{N\gamma\mathfrak{c}_j}(Lx_j) - Lx_j\big)_{k\in\mathscr{C}} = (\xi_{j,k} + \eta)_{k\in\mathscr{C}},$$
$$\text{where} \;\; \eta = \left(J_{N\gamma\mathfrak{c}_j}\left(\sum_{k\in\mathscr{C}} \xi_{j,k}\right) - \sum_{k\in\mathscr{C}} \xi_{j,k}\right)\Big/ N. \tag{22}$$

Example 8 The separable multicommodity flow problem with arc capacity constraints (see, e.g., [3, Section 8.3]) is an instantiation of Example 7 with, for every $j \in \mathscr{A}$, $\mathfrak{c}_j = \partial(\phi_j + \iota_{\Omega_j})$, where $\phi_j \colon \mathbb{R} \to \left]-\infty, +\infty\right]$ is a proper lower semicontinuous convex function and Ω_j is a nonempty closed interval in $\mathbb{R}$ such that $\Omega_j \cap \text{dom}\,\phi_j \neq \varnothing$. In this setting, it follows from [2, Example 23.3 and Proposition 24.47] that

$$(\forall j \in \mathscr{A})\big(\forall \gamma \in \left]0, +\infty\right[\big) \quad J_{\gamma\mathfrak{c}_j} = \text{prox}_{\gamma(\phi_j + \iota_{\Omega_j})} = \text{proj}_{\Omega_j} \circ \text{prox}_{\gamma\phi_j}. \tag{23}$$

Remark 9 Let us compare the realization of (8) in the context of Example 8 to several existing methods for the problem in Example 8.

(i) The method of [13] requires strict convexity of $(\phi_j)_{j\in\mathscr{A}}$, while [16] considers a specific form of $(\phi_j)_{j\in\mathscr{A}}$. We do not have such restrictions.

(ii) The analytic center cutting plane framework (see, e.g., [10], [18, Section 4.3.3], and the references therein) and the algorithms of [11, 13, 15] involve, at every iteration, potentially complex subproblems which have no finite termination guarantee. By contrast, (8) requires the simple subproblem of evaluating $(\text{prox}_{\gamma\phi_j})_{j\in\mathscr{A}}$. As an example, consider the Kleinrock function

$$\phi \colon \mathbb{R} \to \left]-\infty, +\infty\right] : \xi \mapsto \begin{cases} \dfrac{\xi}{\alpha - \xi}, & \text{if } \xi < \alpha; \\ +\infty, & \text{otherwise}, \end{cases} \tag{24}$$

where $\alpha \in \left]0, +\infty\right[$. Then, for every $\gamma \in \left]0, +\infty\right[$ and every $\xi \in \mathbb{R}$, in terms of the variable $s \in \mathbb{R}$, the cubic equation $s^3 - (2\alpha + \xi)s^2 + (\alpha + 2\xi)\alpha s + \alpha(\gamma - \alpha\xi) = 0$ has a unique solution $\overline{s}$ in $\left]-\infty, \alpha\right[$, and $\text{prox}_{\gamma\phi}\xi = \overline{s}$; see also Examples 11–14.

(iii) At every iteration, (8) activates only a subgroup $(\phi_j)_{j\in\mathscr{A}_n}$ of functions, as opposed to all of them as in [11, 13, 15, 16] (see also [18] and the references therein).

(iv) The methods of [10, 16] do not guarantee convergence to a solution of the separable multicommodity flow problem, whereas (8) produces a sequence which converges to a solution.

Remark 10 Consider the standard traffic assignment problem, that is, the special case of Example 7 where $(\forall j \in \mathscr{A})\ C_j = \left[0, +\infty\right[^{\mathscr{C}}$.

(i) In [1, Example 4.4], this problem was solved by an application of the forward-backward method [1, Theorem 2.8], where it is further assumed that, for every $j \in \mathscr{A}$, $\operatorname{dom} \mathfrak{c}_j = \mathbb{R}$ and $\mathfrak{c}_j$ is Lipschitzian. However, some common operators found in the literature of traffic assignment [4] do not fulfill this requirement; their resolvents are provided in Examples 11–14.

(ii) The approach of [14], which is an application of the Douglas–Rachford algorithm [17], requires to compute the projectors onto polyhedral sets of the form $\{(\xi_j)_{j\in\mathscr{A}} \in [0,+\infty[^{\mathscr{A}} \mid (\forall i \in \mathscr{N})\ \ \sum_{j\in\mathscr{A}} \varepsilon_{i,j}\xi_j = \delta_i\}$, where $(\varepsilon_{i,j})_{i\in\mathscr{N},j\in\mathscr{A}}$ are defined in (11). This results in solving a subproblem with strongly convex quadratic cost at every iteration, and there is no closed-form expression for the solution to such problem.

(iii) The more general traffic assignment problem considered in [14], where nonseparable monotone coupling cost operators are used, can be solved via the method of [5, Section 4.1]. However, the realization of the resulting algorithm in the context of the standard traffic assignment problem is not block-iterative in the sense of ③.

Example 11 (Bureau of Public Roads capacity operator) Let $(\alpha, \varrho, \theta, p) \in \left]0,+\infty\right[^4$ and define

$$\mathfrak{c} \colon \mathbb{R} \to \mathbb{R} \colon \xi \mapsto \begin{cases} \theta\left(1+\alpha\left(\dfrac{\xi}{\varrho}\right)^p\right), & \text{if } \xi \geqslant 0;\\ \theta, & \text{if } \xi < 0. \end{cases} \tag{25}$$

In addition, let $\gamma \in \left]0,+\infty\right[$ and $\xi \in \mathbb{R}$. Then the following hold:

(i) Suppose that $\xi \geqslant \gamma\theta$. Then, in terms of the variable $s \in \mathbb{R}$, the equation $\alpha\gamma\theta s^p/\varrho^p + s + \gamma\theta - \xi = 0$ has a unique solution $\overline{s}$ and $J_{\gamma\mathfrak{c}}\xi = \overline{s}$.

(ii) Suppose that $\xi < \gamma\theta$. Then $J_{\gamma\mathfrak{c}}\xi = \xi - \gamma\theta$.

Example 12 (Logarithmic capacity operator) Let $\omega \in \left]0,+\infty\right[$, let $\theta \in \left[0,+\infty\right[$, and define

$$\mathfrak{c} \colon \mathbb{R} \to 2^{\mathbb{R}} \colon \xi \mapsto \begin{cases} \left\{\theta + \ln \dfrac{\omega}{\omega-\xi}\right\}, & \text{if } \xi < \omega;\\ \varnothing, & \text{if } \xi \geqslant \omega. \end{cases} \tag{26}$$

Then

$$\big(\forall\gamma \in \left]0,+\infty\right[\big)(\forall \xi \in \mathbb{R}) \quad J_{\gamma\mathfrak{c}}\xi = \omega - \gamma\mathcal{W}\big(\omega\gamma^{-1}\exp(\theta - \xi/\gamma + \omega/\gamma)\big), \tag{27}$$

where $\mathcal{W}$ is the Lambert W-function, that is, the inverse of $\left[-1,+\infty\right[\to \left[-1/e,+\infty\right[\colon \xi \mapsto \xi\exp(\xi)$.

Example 13 (Traffic Research Corporation capacity operator) Let $(\alpha, \beta, \delta, \omega) \in \left]0,+\infty\right[^4$ and define

$$\mathfrak{c} \colon \mathbb{R} \to \mathbb{R} \colon \xi \mapsto \delta + \alpha(\xi-\omega) + \sqrt{\alpha^2(\xi-\omega)^2+\beta}. \tag{28}$$

Then

$$\big(\forall\gamma \in \left]0,+\infty\right[\big)(\forall \xi \in \mathbb{R}) \quad J_{\gamma\mathfrak{c}}\xi = \frac{-\sqrt{\gamma^2\alpha^2(\xi-\gamma\delta-\omega)^2+(2\gamma\alpha+1)\gamma^2\beta}+\gamma\alpha(\xi-\gamma\delta+\omega)+\xi-\gamma\delta}{2\gamma\alpha+1}. \tag{29}$$

Example 14 Let $\alpha \in]1, +\infty[$, let $\theta \in]0, +\infty[$, let $p \in]0, +\infty[$, and define

$$\mathfrak{c} \colon \mathbb{R} \to \mathbb{R} \colon \xi \mapsto \theta\alpha^{p\xi}. \tag{30}$$

Then

$$\big(\forall \gamma \in]0, +\infty[\big)(\forall \xi \in \mathbb{R}) \quad J_{\gamma\mathfrak{c}}\xi = \xi - \frac{\mathcal{W}\big(\gamma\theta\alpha^{p\xi} p \ln \alpha\big)}{p \ln \alpha}. \tag{31}$$

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