Projecting onto the intersection of a cone and a sphere

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Abstract

The projection onto the intersection of sets generally does not allow for a closed form even when the individual projection operators have explicit descriptions. In this work, we systematically analyze the projection onto the intersection of a cone with either a ball or a sphere. Several cases are provided where the projector is available in closed form. Various examples based on finitely generated cones, the Lorentz cone, and the cone of positive semidefinite matrices are presented. The usefulness of our formulae is illustrated by numerical experiments for determining copositivity of real symmetric matrices.

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1 Introduction

Throughout this paper, we assume that

$$\boxed{\mathcal{H} \text{ is a real Hilbert space}} \tag{1.1}$$

with inner product $\langle \cdot \mid \cdot \rangle$ and induced norm $\|\cdot\|$. Let K and S be subsets of $\mathcal{H}$, with associated projection operator (or projectors)

$$P_K \text{ and } P_S, \tag{1.2}$$

respectively. Our aim is to derive a formula for the projector of the intersection

$$P_{K\cap S}. \tag{1.3}$$

Only in rare cases is it possible to obtain a "closed form" for $P_{K\cap S}$ in terms of P_K and P_S: e.g., when K and S are either both half-spaces (Haugazeau; see [10] and also [3, Corollary 29.25]) or both subspaces (Anderson–Duffin; see [1, Theorem 8] and also [3, Corollary 25.38]). Inspired by an example in the recent and charming book [12], *our aim in this paper is to systematically study the case when K is a closed convex* cone *and S is either the (convex) unit ball or (nonconvex) unit sphere centered at the origin.* In [12, Example 5.5.2], Lange used this projector for an algorithm on determining copositivity of a matrix; however, this projection has the potential to be useful in other settings where, say, *a priori* constraints are present (e.g., positivity and energy). We obtain formulae describing the full (possibly set-valued) projector and also discuss nonpolyhedral cones such as the Lorentz cone or the cone of positive semidefinite matrices. We also revisit Lange's copositivity example and tackle it with other algorithms that appear to perform quite well.

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The remainder of the paper is organized as follows. Section 2 contains miscellaneous results for subsequent use. In Section 3, we provide various results on cones and conical hulls. The description of projections involving cones and subsets of spheres is the topic of Section 4. In Section 5, we turn to results formulated in the Hilbert space of real symmetric matrices. Cones that are finitely generated and corresponding projectors are investigated in Section 6. Our main results are presented in Section 7 (cone intersected with ball) and Section 8 (cone intersected with sphere), respectively. Additional examples are provided in Section 9. In the final Section 10, we put the theory to good use and offer new algorithmic approaches to determine copositivity.

We conclude this introductory section with some comments on notation. For a subset C of $\mathcal{H}$, its *closure* (with respect to the norm topology of $\mathcal{H}$) and *orthogonal complement* are denoted by $\overline{C}$ and $C^{\perp}$, respectively. Next, $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ := [0, +\infty[$, and $\mathbb{R}_{++} :=]0, +\infty[$. In turn, the sphere and the closed ball in $\mathcal{H}$ with center $x \in \mathcal{H}$ and radius $\rho \in \mathbb{R}_{++}$ are respectively defined as $\mathrm{S}(x;\rho) := \{y \in \mathcal{H} \mid \|y - x\| = \rho\}$ and $\mathrm{B}(x;\rho) := \{y \in \mathcal{H} \mid \|y - x\| \leqslant \rho\}$. The product space $\boldsymbol{\mathcal{H}} := \mathcal{H} \oplus \mathbb{R}$ is equipped with the scalar product $((x, \xi), (y, \eta)) \mapsto \langle x \mid y \rangle + \xi\eta$, and we shall use boldface letters for sets and vectors in $\boldsymbol{\mathcal{H}}$. The notation mainly follows [3] or will be introduced as needed.

2 Auxiliary results

In this short section, we collect a few results that will be useful later.

Lemma 2.1 *Let $\{\alpha_i\}_{i \in I}$ be a finite subset of $\mathbb{R}$ such that*

$$(\forall i \in I)\,(\forall j \in I) \quad i \neq j \Rightarrow \alpha_i \alpha_j = 0 \tag{2.1}$$

and that

$$\sum_{i \in I} \alpha_i = 1. \tag{2.2}$$

Then there exists $i \in I$ such that $\alpha_i = 1$ and $(\forall j \in I \smallsetminus \{i\})\ \alpha_j = 0$.

Proof. Suppose that there exist i and j in I such that $i \neq j$, that $\alpha_i \neq 0$, and that $\alpha_j \neq 0$. Then $\alpha_i \alpha_j \neq 0$, which violates (2.1). Hence, $(\alpha_i)_{i \in I}$ contains at most one nonzero number. On the other hand, by (2.2), $(\alpha_i)_{i \in I}$ must contain at least one nonzero number. Altogether, we conclude that there exists $i \in I$ such that $\alpha_i \neq 0$ and $(\forall j \in I \smallsetminus \{i\})\ \alpha_j = 0$. Consequently, it follows from (2.2) that $\alpha_i = 1$, as claimed. ■

Lemma 2.2 *Let $\{x_i\}_{i \in I}$ be a finite subset of $\mathcal{H}$, and let $\{\alpha_i\}_{i \in I}$ be a finite subset of $\mathbb{R}$ such that $\sum_{i \in I} \alpha_i = 1$. Set $x := \sum_{i \in I} \alpha_i x_i$ and $\beta := \|x\|$. Then the following hold:*

(i) $\beta^2 + \sum_{(i,j) \in I \times I} \alpha_i \alpha_j \|x_i - x_j\|^2 / 2 = \sum_{i \in I} \alpha_i \|x_i\|^2$.

(ii) *Suppose that*

$$(\forall i \in I) \quad \|x_i\| = \beta, \tag{2.3}$$

that

$$(\forall i \in I) \quad \alpha_i \geqslant 0, \tag{2.4}$$

and that the vectors $\{x_i\}_{i \in I}$ are pairwise distinct, i.e.,

$$(\forall i \in I)\,(\forall j \in I) \quad i \neq j \Rightarrow x_i \neq x_j. \tag{2.5}$$

Then $(\exists i \in I)\ x = x_i$.

Proof. (i): See, for instance, [3, Lemma 2.14(ii)].

(ii): Since $\sum_{i\in I}\alpha_i = 1$, we deduce from (i) and (2.3) that $\beta^2 + \sum_{(i,j)\in I\times I}\alpha_i\alpha_j\|x_i - x_j\|^2/2 = \sum_{i\in I}\alpha_i\beta^2 = \beta^2$, which yields $\sum_{(i,j)\in I\times I}\alpha_i\alpha_j\|x_i - x_j\|^2 = 0$ or, equivalently, by (2.4),

$$(\forall i \in I)\,(\forall j \in I) \quad \alpha_i\alpha_j\|x_i - x_j\|^2 = 0. \tag{2.6}$$

Thus, we get from (2.5) and (2.6) that $(\forall i \in I)\,(\forall j \in I)\ i \neq j \Rightarrow \|x_i - x_j\| \neq 0 \Rightarrow \alpha_i\alpha_j = 0$, and because $\sum_{i\in I}\alpha_i = 1$, Lemma 2.1 guarantees the existence of $i \in I$ such that $\alpha_i = 1$ and $(\forall j \in I \smallsetminus \{i\})\ \alpha_j = 0$. Consequently, it follows from the very definition of x that $x = x_i$, as desired. ■

Lemma 2.3 *Let α be in $\mathbb{R}$, let β be in $\mathbb{R}_{++}$, and let $\boldsymbol{x} = (x, \xi) \in \boldsymbol{\mathcal{H}}$. Set*[1]

$$\boldsymbol{S}_{\alpha,\beta} := \mathrm{S}(0;\beta) \times \{\alpha\}. \tag{2.7}$$

Then $\max\langle \boldsymbol{x} \mid \boldsymbol{S}_{\alpha,\beta}\rangle = \beta\|x\| + \xi\alpha$.

Proof. We shall assume that $x \neq 0$, since otherwise $\langle \boldsymbol{x} \mid \boldsymbol{S}_{\alpha,\beta}\rangle = \{\xi\alpha\}$ and the assertion is clear. Now, for every $\boldsymbol{y} = (y, \alpha) \in \boldsymbol{S}_{\alpha,\beta}$, since $\|y\| = \beta$, the Cauchy–Schwarz inequality yields

$$\langle \boldsymbol{x} \mid \boldsymbol{y}\rangle = \langle x \mid y\rangle + \xi\alpha \leqslant \|x\|\|y\| + \xi\alpha = \beta\|x\| + \xi\alpha. \tag{2.8}$$

Hence $\sup\langle \boldsymbol{x} \mid \boldsymbol{S}_{\alpha,\beta}\rangle \leqslant \beta\|x\| + \xi\alpha$. Consequently, because $(\beta x/\|x\|, \alpha) \in \boldsymbol{S}_{\alpha,\beta}$ and

$$\left\langle \boldsymbol{x} \,\middle|\, \left(\frac{\beta x}{\|x\|}, \alpha\right)\right\rangle = \left\langle x \,\middle|\, \frac{\beta x}{\|x\|}\right\rangle + \xi\alpha = \beta\|x\| + \xi\alpha, \tag{2.9}$$

we obtain the conclusion. ■

3 Cones and conical hulls

In this section, we systematically study cones and conical hulls.

Let C be a subset of $\mathcal{H}$. Recall that the *convex hull* of C, i.e., the smallest convex subset of $\mathcal{H}$ containing C, is denoted by $\operatorname{conv} C$ and (see, e.g., [3, Proposition 3.4]), it is characterized by

$$\operatorname{conv} C = \left\{ \sum_{i\in I}\alpha_i x_i \,\middle|\, I \text{ is finite, } \{\alpha_i\}_{i\in I} \subseteq \left]0,1\right] \text{ such that } \sum_{i\in I}\alpha_i = 1, \text{ and } \{x_i\}_{i\in I} \subseteq C \right\}. \tag{3.1}$$

Next, C is a *cone* if $C = \bigcup_{\lambda\in\mathbb{R}_{++}}\lambda C$. In turn, the *conical hull* of C is the smallest cone in $\mathcal{H}$ containing C and is denoted by $\operatorname{cone} C$; furthermore, the *closed conical hull* of C, in symbol, $\overline{\operatorname{cone}}\, C$, is the smallest closed cone in $\mathcal{H}$ containing C. Finally, the *polar cone* of C is

$$C^{\ominus} := \{u \in \mathcal{H} \mid \sup\langle u \mid C\rangle \leqslant 0\}, \tag{3.2}$$

and the *recession cone* of C is

$$\operatorname{rec} C := \{x \in \mathcal{H} \mid x + C \subseteq C\}. \tag{3.3}$$

Example 3.1 Let $\rho \in \mathbb{R}_{++}$, and set $C := \mathrm{S}(0;\rho)$. Then $\operatorname{conv} C = \mathrm{B}(0;\rho)$.

Proof. Since $\mathrm{B}(0;\rho)$ is convex and $C \subseteq \mathrm{B}(0;\rho)$, we obtain $\operatorname{conv} C \subseteq \mathrm{B}(0;\rho)$. Conversely, take $x \in \mathrm{B}(0;\rho)$, and we consider the following two conceivable cases:

Case 1: $x = 0$: Fix $y \in C$. Then clearly $-y \in C$ and $x = 0 = (1/2)y + (1/2)(-y) \in \operatorname{conv} C$.

Case 2: $x \neq 0$: Set $x_+ := (\rho/\|x\|)x$, $x_- := (\rho/\|x\|)(-x)$, and $\alpha := (1 + \|x\|/\rho)/2$. Then $\{x_+, x_-\} \subseteq C$, and because $\|x\| \leqslant \rho$, we have $\alpha \in \left]0,1\right]$. Thus, since it is readily verified that $x = \alpha x_+ + (1-\alpha)x_-$, we get $x \in \operatorname{conv} C$.

Hence, $x \in \operatorname{conv} C$ in both cases, which completes the proof. ■

[1] Here and elsewhere, "$\times$" denotes the *Cartesian product* of sets.

For the sake of clarity, let us point out the following.

Remark 3.2 Let K be a nonempty cone in $\mathcal{H}$. Then $0 \in \overline{K}$, and if $K \neq \{0\}$, then $(\forall \rho \in \mathbb{R}_{++})$ $K \cap \mathrm{S}(0;\rho) \neq \varnothing$.

Fact 3.3 *Let C be a subset of $\mathcal{H}$. Then the following hold:*

(i) $\operatorname{cone} C = \bigcup_{\lambda \in \mathbb{R}_{++}} \lambda C$.

(ii) $\overline{\operatorname{cone}}\, C = \overline{\operatorname{cone} C}$.

(iii) $\operatorname{cone}(\operatorname{conv} C)$ *is the smallest convex cone containing C.*

Proof. See, e.g., [3, Proposition 6.2(i)–(iii)]. ■

In general, for subsets C and D of $\mathcal{H}$, $\overline{C \cap D} \neq \overline{C} \cap \overline{D}$. However, the following result provides an interesting instance where taking intersections and closures commutes.

Proposition 3.4 *Let K be a nonempty cone in $\mathcal{H}$, and let $\rho \in \mathbb{R}_{++}$. Then the following hold:*

(i) $\overline{K} \cap \mathrm{S}(0;\rho) = \overline{K \cap \mathrm{S}(0;\rho)}$.

(ii) $\overline{K} \cap \mathrm{B}(0;\rho) = \overline{K \cap \mathrm{B}(0;\rho)}$.

Proof. We assume that

$$K \neq \{0\}\,, \tag{3.4}$$

since otherwise the assertions are clear.

(i): Since we obviously have $\overline{K \cap \mathrm{S}(0;\rho)} \subseteq \overline{K} \cap \mathrm{S}(0;\rho)$, it suffices to verify that $\overline{K} \cap \mathrm{S}(0;\rho) \subseteq \overline{K \cap \mathrm{S}(0;\rho)}$. To do so, take $x \in \overline{K} \cap \mathrm{S}(0;\rho)$. Then, because $x \in \overline{K}$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that

$$x_n \to x. \tag{3.5}$$

In turn, by the continuity of $\|\cdot\|$ and the fact that $x \in \mathrm{S}(0;\rho)$,

$$\|x_n\| \to \|x\| = \rho \in \mathbb{R}_{++}, \tag{3.6}$$

and therefore, we can assume without loss of generality that $(\forall n \in \mathbb{N})$ $\|x_n\| \neq 0$. Hence, for every $n \in \mathbb{N}$, since $x_n \in K$ and $\|\rho x_n/\|x_n\|\| = \rho$, the assumption that K is a cone implies that $\rho x_n/\|x_n\|$ lies in $K \cap \mathrm{S}(0;\rho)$. Thus, $(\rho x_n/\|x_n\|)_{n\in\mathbb{N}}$ is a sequence in $K \cap \mathrm{S}(0;\rho)$; moreover, (3.5) and (3.6) assert that $\rho x_n/\|x_n\| \to \rho x/\rho = x$. Consequently, $x \in \overline{K \cap \mathrm{S}(0;\rho)}$, as announced.

(ii): First, it is clear that $\overline{K \cap \mathrm{B}(0;\rho)} \subseteq \overline{K} \cap \mathrm{B}(0;\rho)$. Conversely, fix $x \in \overline{K} \cap \mathrm{B}(0;\rho)$, and we shall consider two conceivable cases:

(A) $x = 0$: By (3.4), there exists

$$y \in K \smallsetminus \{0\}\,. \tag{3.7}$$

In turn, set

$$(\forall n \in \mathbb{N}) \quad y_n := \frac{\rho}{(n+1)\,\|y\|} y. \tag{3.8}$$

Then, for every $n \in \mathbb{N}$, since K is a cone, (3.7) and (3.8) assert that $y_n \in K$ and thus, since $\|y_n\| = \rho/(n+1) \leqslant \rho$ by (3.8), we deduce that $y_n \in K \cap \mathrm{B}(0;\rho)$. Hence, because $y_n = \rho y/[(n+1)\,\|y\|] \to 0 = x$, we infer that $x \in \overline{K \cap \mathrm{B}(0;\rho)}$.

(B) $x \neq 0$: Since $x \in \overline{K}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that

$$x_n \to x. \tag{3.9}$$

In turn, by the continuity of $\|\cdot\|$,

$$\|x_n\| \to \|x\| \in \mathbb{R}_{++}, \tag{3.10}$$

and we can therefore assume that $(\forall n \in \mathbb{N})\ \|x_n\| \neq 0$. Now set

$$(\forall n \in \mathbb{N}) \quad y_n := \frac{\|x\|}{\|x_n\|} x_n. \tag{3.11}$$

For every $n \in \mathbb{N}$, because $x_n \in K$ and $\|x\| \leqslant \rho$, the assumption that K is a cone and (3.11) yield $y_n \in K \cap \mathrm{B}(0;\rho)$. Consequently, since $y_n \to \|x\| x/\|x\| = x$ due to (3.9) and (3.10), we obtain $x \in \overline{K \cap \mathrm{B}(0;\rho)}$.

To sum up, in both cases, we have $x \in \overline{K \cap \mathrm{B}(0;\rho)}$, and the conclusion follows. ■

We shall require the following notation.

Notation Let C be a nonempty subset of $\mathcal{H}$. Define its *positive span*[2] by

$$\operatorname{pos} C := \left\{ \sum_{i \in I} \alpha_i x_i \;\middle|\; I \text{ is finite, } \{\alpha_i\}_{i \in I} \subseteq \mathbb{R}_+, \text{ and } \{x_i\}_{i \in I} \subseteq C \right\}. \tag{3.12}$$

We observe that if C is finite, then $\operatorname{pos} C$ coincides[3] with the Minkowski sum of the sets $(\mathbb{R}_+ c)_{c \in C}$, i.e.,

$$\operatorname{pos} C = \sum_{c \in C} \mathbb{R}_+ c. \tag{3.13}$$

Lemma 3.5 *Let C be a nonempty subset of $\mathcal{H}$, and set $K := \operatorname{pos} C$. Then the following hold:*

(i) $x \in K^{\ominus} \Leftrightarrow \sup\langle x \mid C\rangle \leqslant 0$.

(ii) $K = \operatorname{cone}(\operatorname{conv}(C \cup \{0\})) = \operatorname{cone}(\{0\} \cup \operatorname{conv} C) = \operatorname{cone}(\operatorname{conv} C) \cup \{0\}$.

(iii) *K is the smallest convex cone containing $C \cup \{0\}$.*

Proof. (i): This follows from the definition of polar cones and (3.12).

(ii): Set $D := \operatorname{cone}(\operatorname{conv}(C \cup \{0\}))$, $E := \operatorname{cone}(\{0\} \cup \operatorname{conv} C)$, and $F := \operatorname{cone}(\operatorname{conv} C) \cup \{0\}$. We shall establish that

$$K \subseteq D \subseteq E = F \subseteq K. \tag{3.14}$$

First, take $x \in K$, say $x = \sum_{i \in I} \alpha_i x_i$, where I is finite, $\{\alpha_i\}_{i \in I} \subseteq \mathbb{R}_+$, and $\{x_i\}_{i \in I} \subseteq C$; in addition, set $\alpha := \sum_{i \in I} \alpha_i$. If $\alpha = 0$, then, because $\{\alpha_i\}_{i \in I} \subseteq \mathbb{R}_+$, we obtain $(\forall i \in I)\ \alpha_i = 0$ and thus $x = 0 \in D$; otherwise, we have $\alpha > 0$ and $x = \alpha \sum_{i \in I} (\alpha_i/\alpha)\, x_i \in \operatorname{cone}(\operatorname{conv} C) \subseteq D$. Next, fix $y \in D$. Then Fact 3.3(i) and (3.1) yield the existence of $\lambda \in \mathbb{R}_{++}$, a finite subset $\{\beta_i\}_{i \in J}$ of $\mathbb{R}_+$, and a finite subset $\{x_i\}_{i \in J}$ of C such that $y = \lambda \sum_{i \in J} \beta_i x_i$. In turn, if $\sum_{i \in J} \beta_i = 0$, then $(\forall i \in J)\ \beta_i = 0$ and so $y = 0 \in E$; otherwise, $\sum_{i \in J} \beta_i > 0$ and, upon setting $\beta := \sum_{i \in J} \beta_i$, we get $y = \lambda\beta \sum_{i \in J} (\beta_i/\beta)\, x_i \in \operatorname{cone}(\operatorname{conv} C) \subseteq E$. Let us now prove that $E = F$. To do so, we infer from Fact 3.3(i) that

$$E = \bigcup_{\lambda \in \mathbb{R}_{++}} \lambda(\{0\} \cup \operatorname{conv} C) = \{0\} \cup \left(\bigcup_{\lambda \in \mathbb{R}_{++}} \lambda \operatorname{conv} C \right) = \{0\} \cup \operatorname{cone}(\operatorname{conv} C) = F. \tag{3.15}$$

Finally, take $z \in F$. If $z = 0$, then clearly $z \in K$; otherwise, $z \in \operatorname{cone}(\operatorname{conv} C)$ and, by Fact 3.3(i) and (3.1), there exist $\mu \in \mathbb{R}_{++}$, a finite subset $\{\delta_i\}_{i \in T}$ of $]0,1]$, and a finite subset $\{x_i\}_{i \in T}$ of C such that $z = \mu \sum_{i \in T} \delta_i x_i = \sum_{i \in T} (\mu\delta_i)\, x_i \in K$. Altogether, (3.14) holds.

(iii): Since $K = \operatorname{cone}(\operatorname{conv}(C \cup \{0\}))$ by (ii), the conclusion thus follows from Fact 3.3(iii). ■

[2] Technically, this should be called the "nonnegative span" but we follow the more common usage.

[3] This readily follows from (3.12).

Example 3.6 (Lorentz cone) Let α and β be in $\mathbb{R}_{++}$, set

$$\boldsymbol{K}_\alpha := \{(x,\xi) \in \boldsymbol{\mathcal{H}} = \mathcal{H} \oplus \mathbb{R} \mid \|x\| \leqslant \alpha\xi\}, \tag{3.16}$$

and set

$$\boldsymbol{C}_{\alpha,\beta} := \mathrm{S}(0;\beta) \times \{\beta/\alpha\} \subseteq \boldsymbol{\mathcal{H}}. \tag{3.17}$$

Then $\boldsymbol{K}_\alpha$ is a nonempty closed convex cone in $\boldsymbol{\mathcal{H}}$ and

$$\boldsymbol{K}_\alpha = \operatorname{pos} \boldsymbol{C}_{\alpha,\beta} = \operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta}) \cup \{\mathbf{0}\}. \tag{3.18}$$

Proof. Since $\boldsymbol{K}_\alpha$ is the epigraph of the function $\|\cdot\|/\alpha$, which is continuous, convex, and positively homogeneous[4], we deduce from [3, Proposition 10.2] that $\boldsymbol{K}_\alpha$ is a nonempty closed convex cone in $\boldsymbol{\mathcal{H}}$. Next, let us establish (3.18). In view of Lemma 3.5(ii), it suffices to show that $\boldsymbol{K}_\alpha = \operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta}) \cup \{\mathbf{0}\}$. Towards this aim, let us first observe that, due to [3, Exercise 3.2] and Example 3.1,

$$\operatorname{conv} \boldsymbol{C}_{\alpha,\beta} = (\operatorname{conv} \mathrm{S}(0;\beta)) \times (\operatorname{conv}\{\beta/\alpha\}) = \mathrm{B}(0;\beta) \times \{\beta/\alpha\}. \tag{3.19}$$

Now set $\boldsymbol{K} := \operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta}) \cup \{\mathbf{0}\}$, and take $\boldsymbol{x} = (x,\xi) \in \boldsymbol{K}_\alpha$. If $\xi = 0$, then (3.16) yields $x = 0$ and so $\boldsymbol{x} = \mathbf{0} \in \boldsymbol{K}$. Otherwise, $\xi > 0$ and we get from (3.16) that $\|\beta(\alpha\xi)^{-1}x\| = \beta(\alpha\xi)^{-1}\|x\| \leqslant \beta$ or, equivalently, $\beta(\alpha\xi)^{-1}x \in \mathrm{B}(0;\beta)$; therefore, it follows from (3.19) that

$$(x,\xi) = \frac{\alpha\xi}{\beta}\left(\frac{\beta}{\alpha\xi}x, \frac{\beta}{\alpha}\right) \in \frac{\alpha\xi}{\beta}(\mathrm{B}(0;\beta) \times \{\beta/\alpha\}) = \frac{\alpha\xi}{\beta} \operatorname{conv} \boldsymbol{C}_{\alpha,\beta} \subseteq \operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta}) \subseteq \boldsymbol{K}. \tag{3.20}$$

Altogether, $\boldsymbol{K}_\alpha \subseteq \boldsymbol{K}$. Conversely, take $\boldsymbol{y} \in \operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta})$. Then, by Fact 3.3(i) and (3.19), there exist $\lambda \in \mathbb{R}_{++}$ and $y \in \mathrm{B}(0;\beta)$ such that $\boldsymbol{y} = \lambda(y, \beta/\alpha) = (\lambda y, \lambda\beta/\alpha)$. In turn, since $\|\lambda y\| \leqslant \lambda\beta = \alpha(\lambda\beta/\alpha)$, we obtain $\boldsymbol{y} \in \boldsymbol{K}_\alpha$. Hence $\operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta}) \subseteq \boldsymbol{K}_\alpha$. This and the fact that $\mathbf{0} \in \boldsymbol{K}_\alpha$ yield $\boldsymbol{K} \subseteq \boldsymbol{K}_\alpha$. Hence $\boldsymbol{K}_\alpha = \boldsymbol{K}$, as claimed. ∎

Here is an improvement of [3, Corollary 6.53].

Proposition 3.7 *Let C be a nonempty closed convex set in $\mathcal{H}$. Suppose that there exists a nonempty closed subset D of C such that $0 \notin D$ and that one of the following holds:*

(a) $(\operatorname{cone} D) \cup \{0\} = (\operatorname{cone} C) \cup \{0\}$.

(b) $\overline{\operatorname{cone}}\, D = \overline{\operatorname{cone}}\, C$.

Then the following hold:

(i) $(\operatorname{cone} C) \cup (\operatorname{rec} C) = \overline{\operatorname{cone}}\, C$.

(ii) *Suppose that* $\operatorname{rec} C = \{0\}$. *Then* $\operatorname{cone}(C \cup \{0\})$ *is closed.*

Proof. Let us first show that (a)⇒(b). To establish this, assume that (a) holds. Then, since $0 \in \overline{\operatorname{cone} D}$ and $0 \in \overline{\operatorname{cone} C}$ due to Remark 3.2, we infer from Fact 3.3(ii) that

$$\overline{\operatorname{cone}}\, D = \overline{\operatorname{cone} D} \cup \{0\} = \overline{(\operatorname{cone} D) \cup \{0\}} = \overline{(\operatorname{cone} C) \cup \{0\}} = \overline{\operatorname{cone} C} \cup \{0\} = \overline{\operatorname{cone}}\, C, \tag{3.21}$$

which verifies the claim. Thus, it is enough to assume that (b) holds and to show that (i)&(ii) hold.

(i): Clearly $\operatorname{cone} C \subseteq \overline{\operatorname{cone}}\, C$. We now prove that $\operatorname{rec} C \subseteq \overline{\operatorname{cone}}\, C$. To this end, take $x \in \operatorname{rec} C$. Then [3, Proposition 6.51] ensures the existence of sequences $(x_n)_{n\in\mathbb{N}}$ in C and $(\alpha_n)_{n\in\mathbb{N}}$ in $]0,1]$ such that $\alpha_n x_n \to x$. Hence, because $\{\alpha_n x_n\}_{n\in\mathbb{N}} \subseteq \operatorname{cone} C$ by Fact 3.3(i), we deduce from Fact 3.3(ii) that $x \in \overline{\operatorname{cone} C} = \overline{\operatorname{cone}}\, C$. Thus $(\operatorname{cone} C) \cup \operatorname{rec} C \subseteq \overline{\operatorname{cone}}\, C$. Conversely, fix $y \in \overline{\operatorname{cone}}\, C = \overline{\operatorname{cone}}\, D$. It then

[4] A function $f\colon \mathcal{H} \to\,]-\infty,+\infty]$ is *positively homogeneous* if $(\forall x \in \mathcal{H})(\forall \lambda \in \mathbb{R}_{++})\ f(\lambda x) = \lambda f(x)$.

follows from Fact 3.3(ii) that $y \in \overline{\operatorname{cone} D}$, and therefore, in view of Fact 3.3(i), there exist sequences $(\beta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{++}$ and $(y_n)_{n\in\mathbb{N}}$ in D such that

$$\beta_n y_n \to y. \tag{3.22}$$

After passing to subsequences and relabeling if necessary, we assume that

$$\beta_n \to \beta \in [0, +\infty]\,. \tag{3.23}$$

In turn, let us establish that $\beta \in \mathbb{R}_+$ by contradiction: assume that $\beta = +\infty$. Then it follows from (3.22) that $\|y_n\| = (1/\beta_n)\,\|\beta_n y_n\| \to 0$ or, equivalently, $y_n \to 0$. Hence, since $\{y_n\}_{n\in\mathbb{N}} \subseteq D$, the closedness of D asserts that $0 \in D$, which violates our assumption. Therefore $\beta \in \mathbb{R}_+$, and this leads to two conceivable cases:

(A) $\beta = 0$: Then, by (3.23), we can assume without loss of generality that $\{\beta_n\}_{n\in\mathbb{N}} \subseteq \left]0,1\right]$. In turn, since $\{y_n\}_{n\in\mathbb{N}} \subseteq D \subseteq C$, we infer from (3.22)&(3.23) and [3, Proposition 6.51] that $y \in \operatorname{rec} C$.

(B) $\beta > 0$: Then, in view of (3.22)&(3.23), $y_n = (1/\beta_n)\,(\beta_n y_n) \to y/\beta$. Therefore, because $\{y_n\}_{n\in\mathbb{N}} \subseteq C$ and C is closed, we obtain $y/\beta \in C$. Consequently, $y \in \beta C \subseteq \operatorname{cone} C$.

To sum up, $(\operatorname{cone} C) \cup (\operatorname{rec} C) = \overline{\operatorname{cone}}\, C$, as announced.

(ii): Since $C = \operatorname{conv} C$ by the convexity of C, we derive from (i) and Lemma 3.5(ii) that $\overline{\operatorname{cone}}\, C = (\operatorname{cone} C) \cup \{0\} = \operatorname{cone}(C \cup \{0\})$, which guarantees that $\operatorname{cone}(C \cup \{0\})$ is closed. ∎

Corollary 3.8 *Let C be a nonempty subset of $\mathcal{H}$, and set $K := \operatorname{pos} C$. Suppose that $0 \notin \operatorname{conv} C$ and that $\operatorname{conv} C$ is weakly compact. Then K is the smallest closed convex cone containing $C \cup \{0\}$.*

Proof. According to Lemma 3.5(iii), it suffices to verify that K is closed. Since $\operatorname{conv} C$ is weakly compact, it is weakly closed and bounded. In turn, on the one hand, since $\operatorname{conv} C$ is convex and weakly closed, we derive from [3, Theorem 3.34] that $\operatorname{conv} C$ is closed. On the other hand, the boundedness of $\operatorname{conv} C$ guarantees that $\operatorname{rec}(\operatorname{conv} C) = \{0\}$. Altogether, because $K = \operatorname{cone}(\{0\} \cup \operatorname{conv} C)$ due to Lemma 3.5(ii) and because $0 \notin \operatorname{conv} C$, applying Proposition 3.7(ii) to $\operatorname{conv} C$ (with the subset D—as in the setting of Proposition 3.7—being $\operatorname{conv} C$) yields the closedness of K, as required. ∎

The following two examples provide instances in which the assumption of Proposition 3.7 holds.

Example 3.9 Let C be a nonempty closed convex subset of $\mathcal{H}$ such that $C \smallsetminus \{0\} \neq \varnothing$. Suppose that there exists $\rho \in \mathbb{R}_{++}$ satisfying

$$(\operatorname{cone} C) \cap \mathrm{S}(0;\rho) \subseteq C, \tag{3.24}$$

and set $D := (\overline{\operatorname{cone}}\, C) \cap \mathrm{S}(0;\rho)$. Then the following hold:

(i) D is a nonempty closed subset of C and $0 \notin D$.

(ii) $(\operatorname{cone} D) \cup \{0\} = (\operatorname{cone} C) \cup \{0\}$.

Proof. (i): The closedness of D is clear. Next, since $0 \notin \mathrm{S}(0;\rho)$, we have $0 \notin D$. In turn, since $C \smallsetminus \{0\} \neq \varnothing$, we see that $\varnothing \neq \overline{\operatorname{cone}}\, C \neq \{0\}$, and since $\overline{\operatorname{cone}}\, C$ is a cone, Remark 3.2 yields $D \neq \varnothing$. Finally, it follows from Fact 3.3(ii), Proposition 3.4(i) (applied to $\operatorname{cone} C$), (3.24), and the closedness of C that $D = \overline{\operatorname{cone} C} \cap \mathrm{S}(0;\rho) = \overline{(\operatorname{cone} C) \cap \mathrm{S}(0;\rho)} \subseteq \overline{C} = C$, as claimed.

(ii): Because $D \subseteq C$, we get $(\operatorname{cone} D) \cup \{0\} \subseteq (\operatorname{cone} C) \cup \{0\}$. Conversely, take $x \in \operatorname{cone} C$. We then deduce from Fact 3.3(i) the existence of $\lambda \in \mathbb{R}_{++}$ and $y \in C$ such that $x = \lambda y$. If $y = 0$, then $x = 0 \in (\operatorname{cone} D) \cup \{0\}$. Otherwise, $\|y\| \neq 0$ and, since $\rho y/\|y\| \in (\operatorname{cone} C) \cap \mathrm{S}(0;\rho) \subseteq D$, we obtain $x = \lambda y = (\lambda\|y\|/\rho)\,(\rho y/\|y\|) \in \operatorname{cone} D$. Therefore $(\operatorname{cone} C) \cup \{0\} \subseteq (\operatorname{cone} D) \cup \{0\}$, and the conclusion follows. ∎

Before we present a new proof of the well-known fact that finitely generated cones are closed (see [17, Theorem 19.1, Corollary 2.6.2 and the remarks following Corollary 2.6.3]), we make a few comments.

Remark 3.10 Let $\{x_i\}_{i\in I}$ be a finite subset of $\mathcal{H}$, and set $C := \operatorname{conv}\{x_i\}_{i\in I}$.

(i) Since $C = \operatorname{conv} \bigcup_{i\in I}\{x_i\}$, [3, Proposition 3.39(i)] implies that C is compact, and so it is closed and bounded. In turn, the boundedness of C gives $\operatorname{rec} C = \{0\}$.

(ii) The geometric interpretation of the proof of Example 3.11 is as follows. If y lies in $C \smallsetminus \{0\}$, then the ray $\mathbb{R}_+ y$ must intersect a "face" of C that does not contain 0.

(iii) Example 3.11 illustrates that the assumption of Proposition 3.7 is mild and covers the case of finitely generated cones.

Example 3.11 Let $\{x_i\}_{i\in I}$ be a finite subset of $\mathcal{H}$ and set

$$K := \sum_{i\in I} \mathbb{R}_+ x_i. \tag{3.25}$$

Then K is the smallest closed convex cone containing $\{x_i\}_{i\in I} \cup \{0\}$.

Proof. We derive from (3.13) and Lemma 3.5(iii) that K is the smallest convex cone in $\mathcal{H}$ containing $\{x_i\}_{i\in I} \cup \{0\}$. Therefore, it suffices to establish the closedness of K. Towards this goal, we first infer from Lemma 3.5(ii) (applied to $\{x_i\}_{i\in I}$) that

$$K = \operatorname{cone}(\{0\} \cup \operatorname{conv}\{x_i\}_{i\in I}). \tag{3.26}$$

Furthermore, we assume that

$$\{x_i\}_{i\in I} \smallsetminus \{0\} \neq \varnothing, \tag{3.27}$$

because otherwise the claim is trivial. In turn, set $C := \operatorname{conv}\{x_i\}_{i\in I}$,

$$\mathcal{I} := \big\{\varnothing \neq J \subseteq I \;\big|\; 0 \notin \operatorname{conv}\{x_i\}_{i\in J}\big\}, \tag{3.28}$$

and

$$D := \bigcup_{J\in\mathcal{I}} \operatorname{conv}\{x_i\}_{i\in J} \subseteq C. \tag{3.29}$$

Then, by (3.27), $\mathcal{I}$ is nonempty,[5] and thus, $0 \notin D \neq \varnothing$. Moreover, D is closed as a finite union of closed sets, namely $(\operatorname{conv}\{x_i\}_{i\in J})_{J\in\mathcal{I}}$. We now claim that

$$(\operatorname{cone} D) \cup \{0\} = (\operatorname{cone} C) \cup \{0\}. \tag{3.30}$$

To do so, it suffices to verify that $(\operatorname{cone} C) \cup \{0\} \subseteq (\operatorname{cone} D) \cup \{0\}$. Take $x \in (\operatorname{cone} C) \smallsetminus \{0\}$. Then Fact 3.3(i) ensures the existence of $\lambda \in \mathbb{R}_{++}$ and

$$y \in C \smallsetminus \{0\} \tag{3.31}$$

such that $x = \lambda y$. Since $y \in C = \operatorname{conv}\{x_i\}_{i\in I}$, there exist a nonempty subset J of I and

$$\{\alpha_i\}_{i\in J} \subseteq \,]0,1] \tag{3.32}$$

such that $\sum_{i\in J} \alpha_i = 1$ and $y = \sum_{i\in J} \alpha_i x_i$. If $J \in \mathcal{I}$, then $y \in \operatorname{conv}\{x_i\}_{i\in J} \subseteq D$ and hence $x = \lambda y \in \operatorname{cone} D$. Otherwise, $0 \in \operatorname{conv}\{x_i\}_{i\in J}$, and there thus exists $\{\beta_i\}_{i\in J} \subseteq [0,1]$ such that $\sum_{i\in J} \beta_i x_i = 0$ and $J_+ := \{i \in J \mid \beta_i > 0\} \neq \varnothing$. In turn, set

$$\gamma := \min_{i\in J_+} \frac{\alpha_i}{\beta_i} \tag{3.33}$$

[5] We just need to pick a nonzero element x_j of $\{x_i\}_{i\in I}$ and set $J := \{j\}$.

and $(\forall i \in J)\ \delta_i := \alpha_i - \gamma\beta_i$. By (3.32) and (3.33),

$$(\forall i \in J) \quad \delta_i \geqslant 0. \tag{3.34}$$

Now fix $j \in J_+$ such that $\alpha_j/\beta_j = \gamma$. Then we get $\delta_j = 0$ as well as $J \smallsetminus \{j\} \neq \varnothing$ (since otherwise, $J = \{j\}$ and $y = \alpha_j x_j = \gamma\beta_j x_j = 0$, which is absurd), and hence,

$$y = y - \gamma 0 = \sum_{i\in J} \alpha_i x_i - \gamma \sum_{i\in J} \beta_i x_i = \sum_{i\in J\smallsetminus\{j\}} \delta_i x_i. \tag{3.35}$$

Therefore, in view of (3.34), (3.35), and (3.31), we must have $\sum_{i\in J\smallsetminus\{j\}} \delta_i > 0$. In turn, if $J \smallsetminus \{j\} \in \mathcal{I}$, then set $\delta := \sum_{i\in J\smallsetminus\{j\}} \delta_i$ and observe that $y = \delta \sum_{i\in J\smallsetminus\{j\}} (\delta_i/\delta) x_i \in \operatorname{cone} D$, which yields $x = \lambda y \in \operatorname{cone} D$. Otherwise, we reapply the procedure to $y = \sum_{i\in J\smallsetminus\{j\}} \delta_i x_i$ recursively until y can be written as $y = \sum_{i\in J'} \gamma_i x_i$, where $J' \in \mathcal{I}$ and $\{\gamma_i\}_{i\in J'} \subseteq \mathbb{R}_+$ satisfying $\mu := \sum_{i\in J'} \gamma_i > 0$. Consequently, $y = \mu \sum_{i\in J'} (\gamma_i/\mu) x_i \in \operatorname{cone} D$, from which we deduce that $x = \lambda y \in \operatorname{cone} D$. Thus (3.30) holds, and since $\operatorname{rec} C = \{0\}$ (see Remark 3.10(i)), it follows from Proposition 3.7(ii) (applied to $C = \operatorname{conv}\{x_i\}_{i\in I}$) and (3.26) that K is closed, as desired. ■

4 Projection operators

Let C be a nonempty subset of $\mathcal{H}$. Recall that its *distance function* is

$$d_C \colon \mathcal{H} \to \mathbb{R} : x \mapsto \inf_{y\in C} \|x - y\| \tag{4.1}$$

while the corresponding *projection operator* (or *projector*) is the set-valued mapping

$$P_C \colon \mathcal{H} \to 2^{\mathcal{H}} : x \mapsto \{u \in C \mid \|x - u\| = d_C(x)\}. \tag{4.2}$$

Furthermore, if C is closed and convex, then, for every $x \in \mathcal{H}$, $P_C x$ is a singleton and we shall identify $P_C x$ with its unique element which is characterized by

$$P_C x \in C \quad \text{and} \quad (\forall y \in C)\ \langle y - P_C x \mid x - P_C x\rangle \leqslant 0; \tag{4.3}$$

see, for instance, [3, Theorem 3.16]. We start by recalling some known results.

Fact 4.1 *Let K be a nonempty closed convex cone in $\mathcal{H}$, and let x and p be in $\mathcal{H}$. Then*

$$p = P_K x \Leftrightarrow \left[\, p \in K,\ x - p \perp p,\ \text{and } x - p \in K^{\ominus} \,\right]. \tag{4.4}$$

Proof. See, e.g., [3, Proposition 6.28]. ■

Let us recall the celebrated Moreau decomposition for cones; see [15].

Fact 4.2 (Moreau) *Let K be a nonempty closed convex cone in $\mathcal{H}$. Then*

$$(\forall x \in \mathcal{H}) \quad x = P_K x + P_{K^{\ominus}} x \quad \text{and} \quad \|x\|^2 = d_K^2(x) + d_{K^{\ominus}}^2(x). \tag{4.5}$$

Lemma 4.3 *Let K be a nonempty closed convex cone in $\mathcal{H}$, and let $x \in \mathcal{H}$. Then the following hold:*

(i) $P_K x \neq 0 \Leftrightarrow x \in \mathcal{H} \smallsetminus K^{\ominus}$.

(ii) *Suppose that $P_K x \neq 0$. Let $\rho \in \mathbb{R}_{++}$, and set $p := (\rho/\|P_K x\|) P_K x$. Then*

$$\|x - p\| = \sqrt{d_K^2(x) + (\|P_K x\| - \rho)^2}. \tag{4.6}$$

Proof. (i): We deduce from Fact 4.2 that $x \in K^{\ominus} \Leftrightarrow x - P_{K^{\ominus}}x = 0 \Leftrightarrow P_K x = 0$, and the claim follows.
(ii): Set $\beta := \rho/\|P_K x\|$. Then, because $x - P_K x \perp P_K x$ by Fact 4.1, the Pythagorean identity implies that

$$\begin{aligned}
\|x-p\|^2 &= \|x - \beta P_K x\|^2 && (4.7a)\\
&= \|(x - P_K x) + (1-\beta)\, P_K x\|^2 && (4.7b)\\
&= \|x - P_K x\|^2 + (1-\beta)^2 \|P_K x\|^2 && (4.7c)\\
&= d_K^2(x) + (1 - \rho/\|P_K x\|)^2 \|P_K x\|^2 && (4.7d)\\
&= d_K^2(x) + (\|P_K x\| - \rho)^2, && (4.7e)
\end{aligned}$$

and thus (4.6) holds. ■

We now turn to projectors onto subsets of spheres.

Lemma 4.4 *Let C be a nonempty subset of $\mathcal{H}$ consisting of vectors of equal norm, let $x \in \mathcal{H}$, and let $p \in C$. Then the following hold*[6]*:*

(i) $p \in P_C x \Leftrightarrow \langle x \mid p\rangle = \max\langle x \mid C\rangle$.

(ii) $P_C x \neq \varnothing$ *if and only if* $\langle x \mid \cdot\,\rangle$ *achieves its supremum over* C.

Proof. (i): Indeed, since $p \in C$ and $(\forall y \in C)\ \|y\| = \|p\|$ by our assumption, we see that

$$\begin{aligned}
p \in P_C x &\Leftrightarrow (\forall y \in C)\ \|x-p\|^2 \leqslant \|x-y\|^2 && (4.8a)\\
&\Leftrightarrow (\forall y \in C)\ -2\langle x \mid p\rangle \leqslant -2\langle x \mid y\rangle && (4.8b)\\
&\Leftrightarrow (\forall y \in C)\ \langle x \mid y\rangle \leqslant \langle x \mid p\rangle && (4.8c)\\
&\Leftrightarrow \langle x \mid p\rangle = \max\langle x \mid C\rangle, && (4.8d)
\end{aligned}$$

which verifies the claim.
(ii): This follows from (i). ■

The following example provides an instance in which $P_C x \neq \varnothing$, where C and x are as in Lemma 4.4.

Example 4.5 Consider the setting of Lemma 4.4 and suppose, in addition, that C is weakly closed. Then $P_C x \neq \varnothing$.

Proof. Since, by assumption, C is bounded and since C is weakly closed, we deduce that C is weakly compact (see, for instance, [3, Lemma 2.36]). Therefore, because $\langle x \mid \cdot\,\rangle$ is weakly continuous, its supremum over C is achieved, and the assertion therefore follows from Lemma 4.4(ii). ■

Lemma 4.6 *Let C be a nonempty subset of $\mathcal{H}$, let $\beta \in \mathbb{R}_{++}$, and let $u \in \operatorname{pos} C$, say $u = \sum_{i\in I} \alpha_i x_i$, where $\{\alpha_i\}_{i\in I}$ and $\{x_i\}_{i\in I}$ are finite subsets of $\mathbb{R}_+$ and C, respectively. Suppose that $\|u\| = \beta$ and that $(\forall y \in C)\ \|y\| = \beta$. Then the following hold:*

(i) $\sum_{i\in I} \alpha_i \geqslant 1$.

(ii) *Let $x \in \mathcal{H}$, and set $\kappa := \sup\langle x \mid C\rangle$. Suppose that $\kappa \in \left]-\infty, 0\right]$ and that $\kappa \leqslant \langle x \mid u\rangle$. Then the following hold:*

  (a) $P_C x \neq \varnothing$ *and* $\langle x \mid u\rangle = \max\langle x \mid C\rangle = \kappa$.

  (b) $u \in \mathrm{S}(0;\beta) \cap \operatorname{cone}(\operatorname{conv} P_C x)$.

[6] The characterization in item (i) plays also a role in [7, Corollaries 2 and 3].

(c) *Suppose that $\kappa < 0$. Then $u \in P_C x$.*

Proof. (i): Since, by assumption, $(\forall i \in I)\ \ \|x_i\| = \beta$, it follows from the triangle inequality that

$$\beta = \|u\| = \left\| \sum_{i \in I} \alpha_i x_i \right\| \leqslant \sum_{i \in I} \alpha_i \|x_i\| = \beta \sum_{i \in I} \alpha_i. \tag{4.9}$$

Therefore, because $\beta > 0$, we obtain $\sum_{i \in I} \alpha_i \geqslant 1$.

(ii): Let us first establish that

$$(\forall i \in I) \quad x_i \in C \smallsetminus P_C x \Rightarrow \alpha_i = 0 \tag{4.10}$$

by contradiction: assume that there exists $i_0 \in I$ such that

$$x_{i_0} \in C \smallsetminus P_C x \tag{4.11}$$

but that

$$\alpha_{i_0} > 0. \tag{4.12}$$

Then, because the vectors in C are of equal norm, we deduce from Lemma 4.4(i) and (4.11) that $\langle x \mid x_{i_0} \rangle < \sup \langle x \mid C \rangle = \kappa$, and so, by (4.12), $\alpha_{i_0} \langle x \mid x_{i_0} \rangle < \alpha_{i_0} \kappa$. Hence, since

$$\begin{cases} \kappa \leqslant 0, \\ \kappa \leqslant \langle x \mid u \rangle, \\ (\forall i \in I)\ 0 \leqslant \alpha_i \text{ and } \langle x \mid x_i \rangle \leqslant \kappa, \end{cases} \tag{4.13}$$

it follows from (i) that

$$\begin{aligned} \kappa \leqslant \langle x \mid u \rangle &= \sum_{i \in I} \alpha_i \langle x \mid x_i \rangle && (4.14a) \\ &= \alpha_{i_0} \langle x \mid x_{i_0} \rangle + \sum_{i \in I \smallsetminus \{i_0\}} \alpha_i \langle x \mid x_i \rangle && (4.14b) \\ &< \alpha_{i_0} \kappa + \sum_{i \in I \smallsetminus \{i_0\}} \alpha_i \kappa && (4.14c) \\ &= \kappa \sum_{i \in I} \alpha_i && (4.14d) \\ &\leqslant \kappa, && (4.14e) \end{aligned}$$

and we thus arrive at a contradiction, namely $\kappa < \kappa$. Therefore, (4.10) holds.

(ii)(a): If $P_C x$ were empty, then (4.10) would yield $(\forall i \in I)\ \alpha_i = 0$ and it would follow that $u = 0$ or, equivalently, $\beta = \|u\| = 0$, which is absurd. Thus $P_C x \neq \varnothing$, and so Lemma 4.4(i) implies that $\kappa = \max \langle x \mid C \rangle$. Furthermore, we infer from (4.13) and (i) that

$$\kappa \leqslant \langle x \mid u \rangle = \sum_{i \in I} \alpha_i \langle x \mid x_i \rangle \leqslant \sum_{i \in I} \alpha_i \kappa = \kappa \sum_{i \in I} \alpha_i \leqslant \kappa, \tag{4.15}$$

and the latter assertion follows.

(ii)(b): In the remainder, since $u \neq 0$, appealing to (4.10), we assume without loss of generality that

$$(\forall i \in I) \quad x_i \in P_C x \tag{4.16}$$

and that $(\forall i \in I)(\forall j \in I)\ i \neq j \Rightarrow x_i \neq x_j$. Hence, upon setting $\alpha := \sum_{i \in I} \alpha_i \geqslant 1$, we deduce from (4.16) that

$$u = \alpha \sum_{i \in I} \frac{\alpha_i}{\alpha} x_i \in \alpha \operatorname{conv} P_C x \subseteq \operatorname{cone}(\operatorname{conv} P_C x). \tag{4.17}$$

Consequently, since $\|u\| = \beta$, the claim follows.

(ii)(c): Invoking Lemma 4.4(i) and (4.16), we get $(\forall i \in I)\ \langle x \mid x_i\rangle = \max\langle x \mid C\rangle = \kappa$. Thus, by (ii)(a), $\kappa = \langle x \mid u\rangle = \sum_{i\in I}\alpha_i\langle x \mid x_i\rangle = \kappa\sum_{i\in I}\alpha_i$, and since $\kappa \neq 0$, it follows that $\sum_{i\in I}\alpha_i = 1$. To summarize, we have

$$\begin{cases} u = \sum_{i\in I}\alpha_i x_i,\\ (\forall i \in I)\ \|x_i\| = \|u\| = \beta,\\ \{\alpha_i\}_{i\in I} \subseteq \mathbb{R}_+ \text{ satisfying } \sum_{i\in I}\alpha_i = 1,\\ (\forall i \in I)(\forall j \in I)\ i \neq j \Rightarrow x_i \neq x_j. \end{cases} \tag{4.18}$$

Lemma 2.2(ii) and (4.16) therefore imply that $(\exists i \in I)\ u = x_i \in P_C x$, as desired. ■

The following example shows that the conclusion of Lemma 4.6(ii)(c) fails if the assumption that $u \in \operatorname{pos} C$ is omitted.

Example 4.7 Suppose that $\mathcal{H} = \mathbb{R}^3$ and that (e_1, e_2, e_3) is the canonical orthonormal basis of $\mathcal{H}$. Set $C := \{e_1, e_2\}$, $x := (-1, -1, 0)$, and $u := (1/2, 1/2, \sqrt{2}/2)$. Then u is not a conical combination of elements of C and, as in the assumption of Lemma 4.6, $(\beta, \kappa) = (1, -1)$. Moreover, a simple computation gives $\|u\| = 1$, $\langle x \mid u\rangle = -1 = \kappa$, and $\|x - e_1\| = \|x - e_2\| = \sqrt{5}$. Hence, $P_C x = C \neq \varnothing$ while $u \notin C$.

5 Projectors onto sets of real symmetric matrices

In this section, N is a strictly positive integer, and suppose that $\mathcal{H} = \mathbb{S}^N$ is the Hilbert space of real symmetric matrices endowed with the scalar product $\langle\cdot \mid \cdot\rangle\colon (A, B) \mapsto \operatorname{tra}(AB)$, where tra is the trace function; the associated norm is the Frobenius norm $\|\cdot\|_{\mathsf{F}}$. The closed convex cone of positive semidefinite symmetric matrix in $\mathcal{H}$ is denoted by $\mathbb{S}^N_+$, and the set of orthogonal matrices of size $N \times N$ is $\mathbb{U}^N := \{U \in \mathbb{R}^{N\times N} \mid UU^{\mathsf{T}} = \mathrm{Id}\}$, where Id is the identity matrix of $\mathbb{R}^{N\times N}$. Next, for every $x = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$, set $x_+ := (\max\{\xi_i, 0\})_{1\leqslant i\leqslant N}$ and define $\operatorname{Diag} x$ to be the diagonal matrix whose, starting from the upper left corner, diagonal entries are $\xi_1, \ldots, \xi_N$. Now, for every $A \in \mathcal{H}$, the eigenvalues of A (not necessarily distinct) are denoted by $(\lambda_i(A))_{1\leqslant i\leqslant N}$ with the convention that $\lambda_1(A) \geqslant \cdots \geqslant \lambda_N(A)$. In turn, the mapping $\lambda\colon \mathcal{H} \to \mathbb{R}^N : A \mapsto (\lambda_1(A), \ldots, \lambda_N(A))$ is well defined. Finally, the Euclidean scalar product and norm of $\mathbb{R}^N$ are respectively denoted by $\langle\cdot \mid \cdot\rangle$ and $\|\cdot\|$.

Remark 5.1 Let $A \in \mathcal{H}$, $U \in \mathbb{U}^N$, and $x \in \mathbb{R}^N$. Then it is straightforward to verify that

$$\|UAU^{\mathsf{T}}\|_{\mathsf{F}} = \|A\|_{\mathsf{F}} = \|\lambda(A)\| \tag{5.1}$$

and that

$$\|U(\operatorname{Diag} x)U^{\mathsf{T}}\|_{\mathsf{F}} = \|\operatorname{Diag} x\|_{\mathsf{F}} = \|x\|. \tag{5.2}$$

Lemma 5.2 *Set $K := \mathbb{S}^N_+$. Let $A \in \mathcal{H}$, and let $U \in \mathbb{U}^N$ be such that $A = U(\operatorname{Diag}\lambda(A))U^{\mathsf{T}}$. Then $P_K A = U(\operatorname{Diag}(\lambda(A))_+)U^{\mathsf{T}}$ and $\|P_K A\|_{\mathsf{F}} = \|(\lambda(A))_+\|$.*

Proof. It is well known that $P_K A = U(\operatorname{Diag}(\lambda(A))_+)U^{\mathsf{T}}$ (see, e.g., [13, Theorem A1] or [3, Example 29.32]). In turn, since $U \in \mathbb{U}^N$, it follows from Remark 5.1 that $\|P_K A\|_{\mathsf{F}} = \|(\lambda(A))_+\|$. ■

Fact 5.3 (Theobald) (See [18].) *Let A and B be in $\mathcal{H}$. Then the following hold:*

(i) $\langle A \mid B\rangle \leqslant \langle\lambda(A) \mid \lambda(B)\rangle$.

(ii) $\langle A \mid B\rangle = \langle\lambda(A) \mid \lambda(B)\rangle$ *if and only if there exists $U \in \mathbb{U}^N$ such that $A = U(\operatorname{Diag}\lambda(A))U^{\mathsf{T}}$ and $B = U(\operatorname{Diag}\lambda(B))U^{\mathsf{T}}$.*

Lemma 5.4 *Let $\rho \in \mathbb{R}_{++}$, and set*

$$\mathsf{C}_\rho := \Big\{ A \in \mathbb{S}^N_+ \;\Big|\; \operatorname{rank} A = 1 \text{ and } \|A\|_{\mathsf{F}} = \rho \Big\}. \tag{5.3}$$

Then the following hold:

(i) $\mathbb{S}^N_+ = \operatorname{pos} \mathsf{C}_\rho$.

(ii) $\mathsf{C}_\rho = \{A \in \mathcal{H} \mid (\exists U \in \mathbb{U}^N)\ A = U(\operatorname{Diag}(\rho, 0, \ldots, 0))U^\mathsf{T}\}$.

(iii) *Let $A \in \mathcal{H}$. Then $\max\langle A \mid \mathsf{C}_\rho\rangle = \rho\lambda_1(A)$ and*

$$P_{\mathsf{C}_\rho} A = \Big\{ U(\operatorname{Diag}(\rho, 0, \ldots, 0))U^\mathsf{T} \;\Big|\; U \in \mathbb{U}^N \text{ such that } A = U(\operatorname{Diag}\lambda(A))U^\mathsf{T} \Big\} \neq \varnothing. \tag{5.4}$$

Proof. (i): Set $I := \{1, \ldots, N\}$, and let $(e_i)_{i \in I}$ be the canonical orthonormal basis of $\mathbb{R}^N$. First, since $\mathsf{C}_\rho \cup \{0\} \subseteq \mathbb{S}^N_+$ and $\mathbb{S}^N_+$ is a convex cone, we infer from Lemma 3.5(iii) that $\operatorname{pos} \mathsf{C}_\rho \subseteq \mathbb{S}^N_+$. Conversely, take $A \in \mathbb{S}^N_+$, and let $U \in \mathbb{U}^N$ be such that $A = U(\operatorname{Diag}\lambda(A))U^\mathsf{T}$; in addition, set $(\forall i \in I)\ D_i := \operatorname{Diag}(\rho e_i) \in \mathbb{S}^N_+$. Then, for every $i \in I$, since $\operatorname{rank} D_i = 1$ and $\|UD_iU^\mathsf{T}\|_\mathsf{F} = \|D_i\|_\mathsf{F} = \|\rho e_i\| = \rho$, we get from (5.3) that $UD_iU^\mathsf{T} \in \mathsf{C}_\rho$. In turn, because $\{\lambda_i(A)\}_{i \in I} \subseteq \mathbb{R}_+$ and

$$A = U(\operatorname{Diag}\lambda(A))U^\mathsf{T} = U\left(\sum_{i \in I} \operatorname{Diag}(\lambda_i(A)e_i)\right)U^\mathsf{T} = U\left(\sum_{i \in I} \frac{\lambda_i(A)}{\rho} D_i\right)U^\mathsf{T} = \sum_{i \in I} \frac{\lambda_i(A)}{\rho}(UD_iU^\mathsf{T}), \tag{5.5}$$

we deduce that $A \in \operatorname{pos}\mathsf{C}_\rho$. Hence, $\mathbb{S}^N_+ = \operatorname{pos}\mathsf{C}_\rho$.

(ii): Recall that, if A is a matrix of rank r in $\mathbb{S}^N_+$, then

$$\lambda_1(A) \geqslant \cdots \geqslant \lambda_r(A) > \lambda_{r+1}(A) = \cdots \lambda_N(A) = 0. \tag{5.6}$$

Now set $\mathsf{D} := \{A \in \mathcal{H} \mid (\exists U \in \mathbb{U}^N)\ A = U(\operatorname{Diag}(\rho, 0, \ldots, 0))U^\mathsf{T}\}$. First, take $A \in \mathsf{C}_\rho$, and let $U \in \mathbb{U}^N$ be such that $A = U(\operatorname{Diag}\lambda(A))U^\mathsf{T}$. Then, since $\operatorname{rank} A = 1$ and $A \in \mathbb{S}^N_+$, it follows from (5.6) that $\lambda(A) = (\lambda_1(A), 0, \ldots, 0)$ and $\lambda_1(A) > 0$; therefore, because $\|A\|_\mathsf{F} = \rho$, we obtain $\rho = \|A\|_\mathsf{F} = \|\lambda(A)\| = \lambda_1(A)$. Hence, $A = U(\operatorname{Diag}(\lambda_1(A), 0, \ldots, 0))U^\mathsf{T} = U(\operatorname{Diag}(\rho, 0, \ldots, 0))U^\mathsf{T}$, which yields $A \in \mathsf{D}$. Conversely, take $B \in \mathsf{D}$, say $B = V(\operatorname{Diag}(\rho, 0, \ldots, 0))V^\mathsf{T}$, where $V \in \mathbb{U}^N$. Then, since $\rho > 0$, we have $B \in \mathbb{S}^N_+$. Next, on the one hand, because V is nonsingular and $\rho \neq 0$, we have $\operatorname{rank} B = \operatorname{rank}\operatorname{Diag}(\rho, 0, \ldots, 0) = 1$. On the other hand, since $V \in \mathbb{U}^N$, it follows that $\|B\|_\mathsf{F} = \|V(\operatorname{Diag}(\rho, 0, \ldots, 0))V^\mathsf{T}\|_\mathsf{F} = \|(\rho, 0, \ldots, 0)\| = \rho$. Altogether, $B \in \mathsf{C}_\rho$, which completes the proof.

(iii): First, it follows from (ii) that

$$(\forall B \in \mathcal{H}) \quad B \in \mathsf{C}_\rho \Leftrightarrow \lambda(B) = (\rho, 0, \ldots, 0). \tag{5.7}$$

Next, denote the right-hand set of (5.4) by D. Then, by (ii), $\varnothing \neq \mathsf{D} \subseteq \mathsf{C}_\rho$. Now, for every $B \in \mathsf{C}_\rho$, since $\lambda(B) = (\rho, 0, \ldots, 0)$, we infer from Fact 5.3(i) that $\langle A \mid B\rangle \leqslant \langle \lambda(A) \mid \lambda(B)\rangle = \rho\lambda_1(A)$. Thus, $\sup\langle A \mid \mathsf{C}_\rho\rangle \leqslant \rho\lambda_1(A)$. Furthermore, by (5.7), Fact 5.3(ii), and the very definition of D, we see that

$$(\forall B \in \mathsf{C}_\rho) \quad \langle A \mid B\rangle = \rho\lambda_1(A) \Leftrightarrow \langle A \mid B\rangle = \langle \lambda(A) \mid \lambda(B)\rangle \tag{5.8a}$$

$$\Leftrightarrow (\exists U \in \mathbb{U}^N) \begin{cases} A = U(\operatorname{Diag}\lambda(A))U^\mathsf{T}, \\ B = U(\operatorname{Diag}\lambda(B))U^\mathsf{T} \end{cases} \tag{5.8b}$$

$$\Leftrightarrow (\exists U \in \mathbb{U}^N) \begin{cases} A = U(\operatorname{Diag}\lambda(A))U^\mathsf{T}, \\ B = U(\operatorname{Diag}(\rho, 0, \ldots, 0))U^\mathsf{T} \end{cases} \tag{5.8c}$$

$$\Leftrightarrow B \in \mathsf{D}. \tag{5.8d}$$

Therefore, because $\mathsf{D} \neq \varnothing$, we deduce that $\max\langle A \mid \mathsf{C}_\rho\rangle = \rho\lambda_1(A)$ and $(\forall B \in \mathsf{C}_\rho)\ \langle A \mid B\rangle = \max\langle A \mid \mathsf{C}_\rho\rangle \Leftrightarrow B \in \mathsf{D}$. Consequently, since the matrices in C_ρ are of equal norm by (5.3), we derive from Lemma 4.4(i) that $P_{\mathsf{C}_\rho} A = \mathsf{D}$, as desired. ■

6 Projectors onto cones generated by orthonormal sets

We start with a conical version of [3, Example 3.10].

Theorem 6.1 *Let $\{e_i\}_{i\in I}$ be a nonempty finite orthonormal subset of $\mathcal{H}$, set*

$$K := \sum_{i\in I} \mathbb{R}_+ e_i, \tag{6.1}$$

and let $x \in \mathcal{H}$. Then K is a nonempty closed convex cone in $\mathcal{H}$,

$$P_K x = \sum_{i\in I} \max\{\langle x \,|\, e_i\rangle, 0\}\, e_i, \quad \text{and} \quad d_K(x) = \sqrt{\|x\|^2 - \sum_{i\in I}(\max\{\langle x \,|\, e_i\rangle, 0\})^2}. \tag{6.2}$$

Proof. We first infer from Example 3.11 that K is a nonempty closed convex cone. Thus, it is enough to verify (6.2). To this end, set

$$(\forall i \in I) \quad \alpha_i := \max\{\langle x \,|\, e_i\rangle, 0\} \in \mathbb{R}_+ \tag{6.3}$$

and

$$p := \sum_{i\in I} \alpha_i e_i. \tag{6.4}$$

Then, by (6.3)&(6.4)&(6.1), we have $p \in K$, and by assumption, we get

$$\|p\|^2 = \left\|\sum_{i\in I} \alpha_i e_i\right\|^2 = \sum_{i\in I} \alpha_i^2. \tag{6.5}$$

Furthermore, (6.3) implies that

$$(\forall i \in I) \quad [\, \alpha_i = \langle x \,|\, e_i\rangle \text{ or } \alpha_i = 0 \,] \Leftrightarrow \alpha_i(\langle x \,|\, e_i\rangle - \alpha_i) = 0 \Leftrightarrow \alpha_i\langle x \,|\, e_i\rangle = \alpha_i^2, \tag{6.6}$$

and therefore, we get from (6.4) that

$$\langle x \,|\, p\rangle = \left\langle x \,\middle|\, \sum_{i\in I} \alpha_i e_i \right\rangle = \sum_{i\in I} \alpha_i \langle x \,|\, e_i\rangle = \sum_{i\in I} \alpha_i^2. \tag{6.7}$$

In turn, on the one hand, (6.5) and (6.7) yield $\langle x - p \,|\, p\rangle = \langle x \,|\, p\rangle - \|p\|^2 = 0$. On the other hand, invoking (6.4), (6.3), and our hypothesis, we deduce that

$$(\forall i \in I) \quad \langle x - p \,|\, e_i\rangle = \langle x \,|\, e_i\rangle - \left\langle \sum_{j\in I} \alpha_j e_j \,\middle|\, e_i \right\rangle = \langle x \,|\, e_i\rangle - \alpha_i \leqslant 0, \tag{6.8}$$

and hence, by (6.1), $x - p \in K^{\ominus}$. Altogether, we conclude that $P_K x = p = \sum_{i\in I} \max\{\langle x \,|\, e_i\rangle, 0\}\, e_i$ via Fact 4.1. Consequently, (6.5)&(6.7)&(6.3) give

$$d_K^2(x) = \|x - p\|^2 = \|x\|^2 - 2\langle x \,|\, p\rangle + \|p\|^2 = \|x\|^2 - \sum_{i\in I} \alpha_i^2 = \|x\|^2 - \sum_{i\in I}(\max\{\langle x \,|\, e_i\rangle, 0\})^2, \tag{6.9}$$

which completes the proof. ■

Remark 6.2 Here are a few comments concerning Theorem 6.1.

(i) In the setting of Theorem 6.1, suppose that $\{e_i\}_{i\in I}$ is a singleton, say e. Then $K = \mathbb{R}_+ e$ is a ray and (6.2) becomes

$$P_K x = \max\{\langle x \,|\, e\rangle, 0\}\, e \quad \text{and} \quad d_K(x) = \sqrt{\|x\|^2 - (\max\{\langle x \,|\, e\rangle, 0\})^2}, \tag{6.10}$$

which is precisely the formula for projectors onto rays (see, e.g., [3, Example 29.31]).

(ii) Consider the setting of Theorem 6.1. Suppose that N is a strictly positive integer, that $I = \{1,\ldots,N\}$, that $\mathcal{H} = \mathbb{R}^N$, and that $(e_i)_{i\in I}$ is the canonical orthonormal basis of $\mathcal{H}$. Then $K = \mathbb{R}^N_+$ is the positive orthant in $\mathcal{H}$. Now take $x = (\xi_i)_{i\in I} \in \mathcal{H}$. In the light of (6.2), since $(\forall i \in I)\ \langle x \mid e_i\rangle = \xi_i$, we retrieve the well-known formula

$$P_K x = (\max\{\xi_i, 0\})_{i\in I}; \tag{6.11}$$

see, for instance, [3, Example 6.29]. Moreover, upon setting $I_- := \{i \in I \mid \xi_i < 0\}$, we derive from (6.2) that

$$d_K(x) = \sqrt{\|x\|^2 - \sum_{i\in I}(\max\{\xi_i, 0\})^2} = \sqrt{\sum_{i\in I}\xi_i^2 - \sum_{i\in I\smallsetminus I_-}\xi_i^2} = \sqrt{\sum_{i\in I_-}\xi_i^2} \tag{6.12}$$

with the convention that $\sum_{i\in\varnothing}\xi_i^2 = 0$.

Corollary 6.3 *Let $\{e_i\}_{i\in I}$ be a nonempty finite orthonormal subset of $\mathcal{H}$. Set*

$$K := \{y \in \mathcal{H} \mid (\forall i \in I)\ \langle y \mid e_i\rangle \leqslant 0\}, \tag{6.13}$$

and let $x \in \mathcal{H}$. Then K is a nonempty closed convex cone in $\mathcal{H}$,

$$P_K x = x - \sum_{i\in I}\max\{\langle x \mid e_i\rangle, 0\}\, e_i, \quad \textit{and} \quad d_K(x) = \sqrt{\sum_{i\in I}(\max\{\langle x \mid e_i\rangle, 0\})^2}. \tag{6.14}$$

Proof. Since

$$K = \bigcap_{i\in I}\{e_i\}^{\ominus}, \tag{6.15}$$

we see that K is a nonempty closed convex cone. Next, by (6.15), [3, Proposition 6.27] implies that $K = \bigcap_{i\in I}(\mathbb{R}_+ e_i)^{\ominus} = (\sum_{i\in I}\mathbb{R}_+ e_i)^{\ominus}$, and since $\sum_{i\in I}\mathbb{R}_+ e_i$ is a nonempty closed convex cone by Example 3.11, taking the polar cones and invoking [3, Corollary 6.34] yield $K^{\ominus} = (\sum_{i\in I}\mathbb{R}_+ e_i)^{\ominus\ominus} = \sum_{i\in I}\mathbb{R}_+ e_i$. Hence, according to Moreau's theorem (Fact 4.2) and Theorem 6.1, we conclude that $P_K x = x - P_{K^{\ominus}} x = x - \sum_{i\in I}\max\{\langle x \mid e_i\rangle, 0\}\, e_i$ and that

$$d_K^2(x) = \|x\|^2 - d_{K^{\ominus}}^2(x) = \|x\|^2 - \left(\|x\|^2 - \sum_{i\in I}(\max\{\langle x \mid e_i\rangle, 0\})^2\right) = \sum_{i\in I}(\max\{\langle x \mid e_i\rangle, 0\})^2, \tag{6.16}$$

as claimed in (6.14). ∎

7 The projector onto the intersection of a cone and a ball

Our first set of main results is presented in this section. It turns out that the projector onto the intersection of a cone and a ball has a pleasing explicit form.

Theorem 7.1 (cone intersected with ball) *Let K be a nonempty closed convex cone in $\mathcal{H}$, let $\rho \in \mathbb{R}_{++}$, and set $C := K \cap \mathrm{B}(0;\rho)$. Then*

$$(\forall x \in \mathcal{H}) \quad P_C x = \frac{\rho}{\max\{\|P_K x\|, \rho\}} P_K x \quad \textit{and} \quad d_C(x) = \sqrt{d_K^2(x) + (\max\{\|P_K x\| - \rho, 0\})^2}. \tag{7.1}$$

Proof. Take $x \in \mathcal{H}$, set $\beta := \rho / \max\{\|P_K x\|, \rho\} \in \mathbb{R}_{++}$, and set $p := \beta P_K x$. Then, since K is a cone and $P_K x \in K$, we get $p \in K$, and thus, since $\|p\| = \beta\|P_K x\| = \rho(\|P_K x\| / \max\{\|P_K x\|, \rho\}) \leqslant \rho$, it follows that $p \in K \cap \mathrm{B}(0;\rho) = C$. Hence, because C is closed and convex, in the light of (4.3), it remains to verify that $(\forall y \in C)\ \langle x - p \mid y - p\rangle \leqslant 0$. To this end, take $y \in C$, and we consider two alternatives:

(A) $\|P_K x\| \leqslant \rho$: Then $\beta = \rho/\rho = 1$. It follows that $p = P_K x$, and so

$$\|x - p\| = \|x - P_K x\| = d_K(x). \tag{7.2}$$

Next, because $y \in K$, (4.3) asserts that $\langle x - p \mid y - p \rangle = \langle x - P_K x \mid y - P_K x \rangle \leqslant 0$.

(B) $\|P_K x\| > \rho$: Then $\beta = \rho/\|P_K x\| \in \left]0,1\right[$, and so Lemma 4.3(ii) implies that

$$\|x - p\| = \sqrt{d_K^2(x) + (\|P_K x\| - \rho)^2}. \tag{7.3}$$

In turn, on the one hand, since y belongs to the cone K, it follows that $(1/\beta)\,y \in K$, from which and (4.3) we deduce that

$$\langle x - P_K x \mid y - \beta P_K x \rangle = \beta \langle x - P_K x \mid (1/\beta)\,y - P_K x \rangle \leqslant 0. \tag{7.4}$$

On the other hand, because $y \in \mathrm{B}(0;\rho)$ and $\beta = \rho/\|P_K x\|$, the Cauchy–Schwarz inequality yields

$$\langle P_K x \mid y - \beta P_K x \rangle = \langle P_K x \mid y \rangle - \rho\|P_K x\| \leqslant \|P_K x\|\|y\| - \rho\|P_K x\| \leqslant 0. \tag{7.5}$$

Altogether, combining (7.4)&(7.5) and using the fact that $\beta \in \left]0,1\right[$, we obtain

$$\langle x - p \mid y - p \rangle = \langle x - \beta P_K x \mid y - \beta P_K x \rangle \tag{7.6a}$$
$$= \langle x - P_K x \mid y - \beta P_K x \rangle + (1-\beta)\,\langle P_K x \mid y - \beta P_K x \rangle \tag{7.6b}$$
$$\leqslant 0. \tag{7.6c}$$

Hence, in both cases, we have $\langle x - p \mid y - p \rangle \leqslant 0$. Thus $p = P_C x$, and it follows from (7.2)&(7.3) that

$$d_C(x) = \|x - P_C x\| = \|x - p\| = \sqrt{d_K^2(x) + \big(\max\{\|P_K x\| - \rho, 0\}\big)^2}, \tag{7.7}$$

as stated in (7.1). ∎

Here are some easy consequences of Theorem 7.1.

Example 7.2 In the setting of Theorem 7.1, suppose that $K = \mathcal{H}$. Then $C = \mathrm{B}(0;\rho)$, $P_K = \mathrm{Id}$, $d_K \equiv 0$, and (7.1) becomes

$$(\forall x \in \mathcal{H}) \quad P_C x = \frac{\rho}{\max\{\|x\|, \rho\}} x \quad \text{and} \quad d_C(x) = \max\{\|x\| - \rho, 0\}. \tag{7.8}$$

We thus recover the formula for projectors onto balls.

Corollary 7.3 *Let K be a nonempty closed convex cone in $\mathcal{H}$, let $\rho \in \mathbb{R}_{++}$, and set $C := K \cap \mathrm{B}(0;\rho)$. Then*[7] *$P_C = P_{\mathrm{B}(0;\rho)} \circ P_K$.*

Proof. Combine (7.1) and (7.8). Alternatively, set[8] $f := \iota_{\mathrm{B}(0;\rho)}$ and $\kappa := \iota_K$ in the equivalence (iii)⇔(iv) of [19, Theorem 4]. (Note that $\iota_{\mathrm{B}(0;\rho)} + \iota_K = \iota_C$.) ∎

Remark 7.4 In the setting of Corollary 7.3, as we shall see in Example 7.5, $P_C \neq P_K \circ P_{\mathrm{B}(0;\rho)}$, i.e., $P_{\mathrm{B}(0;\rho)} \circ P_K \neq P_K \circ P_{\mathrm{B}(0;\rho)}$, in general.

[7] Here and elsewhere, "$\circ$" denotes the composition of operators.

[8] We use the symbol ι_C to denote the *indicator function* of a subset C of $\mathcal{H}$: $\iota_C(x) = 0$, if $x \in C$; $\iota_C(x) = +\infty$, if $x \notin C$.

Example 7.5 Suppose that $\mathcal{H} = \mathbb{R}^2$. Set $K := \mathbb{R}^2_+$ and $x := (1, -1)$. Then (see also Figure 1)

$$(P_K \circ P_{\mathrm{B}(0;1)})x = P_K(P_{\mathrm{B}(0;1)}x) \overset{(7.8)}{=} P_K\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \overset{(6.11)}{=} \left(\frac{1}{\sqrt{2}}, 0\right) \tag{7.9}$$

and

$$(P_{\mathrm{B}(0;1)} \circ P_K)x = P_{\mathrm{B}(0;1)}(P_K x) \overset{(6.11)}{=} P_{\mathrm{B}(0;1)}(1,0) \overset{(7.8)}{=} (1,0). \tag{7.10}$$

Hence

$$P_{\mathrm{B}(0;1)} \circ P_K \neq P_K \circ P_{\mathrm{B}(0;1)}. \tag{7.11}$$

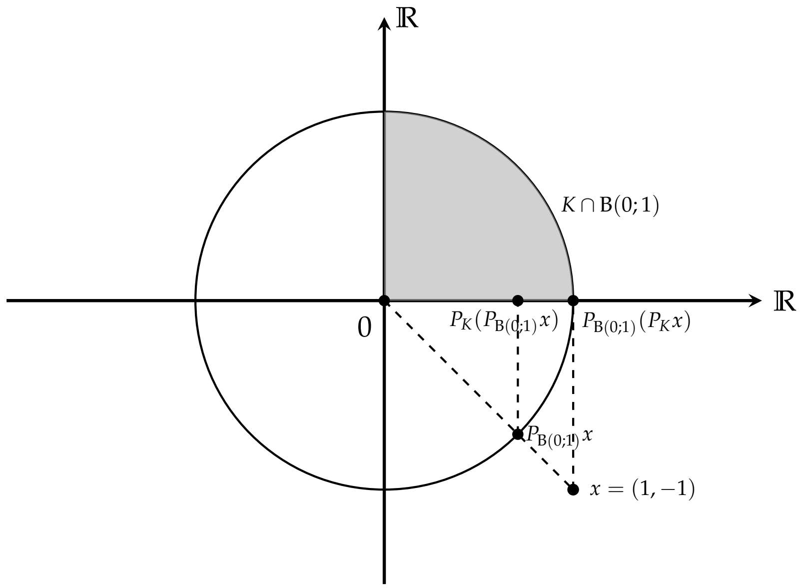

Figure 1 Example 7.5 illustrates that the projectors onto a cone and ball may fail to commute.

As will be seen in the next result, Example 7.5 is, however, not a coincidence.

Corollary 7.6 *Let K be a nonempty closed convex cone in $\mathcal{H}$, and let $\rho \in \mathbb{R}_{++}$. Then*

$$(\forall x \in \mathcal{H}) \quad (P_K \circ P_{\mathrm{B}(0;\rho)})x = \frac{\rho}{\max\{\|x\|, \rho\}} P_K x. \tag{7.12}$$

Proof. It follows from (7.8) and [3, Proposition 29.29] that

$$(\forall x \in \mathcal{H}) \quad (P_K \circ P_{\mathrm{B}(0;\rho)})x = P_K(P_{\mathrm{B}(0;\rho)}x) = P_K\left(\frac{\rho}{\max\{\|x\|, \rho\}} x\right) = \frac{\rho}{\max\{\|x\|, \rho\}} P_K x, \tag{7.13}$$

as desired. ■

Remark 7.7 Consider the setting of Corollary 7.6. Using Corollary 7.6, Theorem 7.1, and Corollary 7.3, we deduce that

$$(\forall x \in \mathcal{H}) \quad (P_K \circ P_{\mathrm{B}(0;\rho)})x = \frac{\max\{\|P_K x\|, \rho\}}{\max\{\|x\|, \rho\}} (P_{\mathrm{B}(0;\rho)} \circ P_K)x. \tag{7.14}$$

8 The projector onto the intersection of a cone and a sphere

In this section, which contains our second half of main results, we develop formulae for the projector onto the intersection of a cone and a sphere.

Theorem 8.1 *Let K be a nonempty closed convex cone in $\mathcal{H}$, let $\rho \in \mathbb{R}_{++}$, and set $C := K \cap \mathrm{S}(0;\rho)$. Suppose that $K \neq \{0\}$. Then the following hold:*

(i) $(\forall x \in K^{\perp})$ $P_C x = C$ *and* $d_C(x) = \sqrt{\|x\|^2 + \rho^2}$.

(ii) $(\forall x \in \mathcal{H} \smallsetminus K^{\ominus})$ $P_C x = \{(\rho/\|P_K x\|)P_K x\}$ *and* $d_C(x) = \sqrt{d_K^2(x) + (\|P_K x\| - \rho)^2}$.

Proof. We first observe that, by assumption and Remark 3.2, $C \neq \varnothing$.

(i): Fix $x \in K^{\perp}$. Then, for every $y \in C = K \cap \mathrm{S}(0;\rho)$, since $x \perp y$ and $\|y\| = \rho$, we get $\|x - y\|^2 = \|x\|^2 + \|y\|^2 = \|x\|^2 + \rho^2$. It follows that $d_C(x) = \sqrt{\|x\|^2 + \rho^2}$ and that $P_C x = C$, as desired.

(ii): First, by the very definition of C, we see that

$$C \text{ consists of vectors of equal norm.} \tag{8.1}$$

Now take $x \in \mathcal{H} \smallsetminus K^{\ominus}$; set[9] $\alpha := \rho/\|P_K x\| \in \mathbb{R}_{++}$ and

$$p := \alpha P_K x. \tag{8.2}$$

Then, because $P_K x$ belongs to the cone K, we obtain $p \in K$, and because

$$\|p\| = \left\| \frac{\rho}{\|P_K x\|} P_K x \right\| = \rho, \tag{8.3}$$

it follows that

$$p \in K \cap \mathrm{S}(0;\rho) = C. \tag{8.4}$$

Next, fix $y \in C$. Since $y \in C \subseteq K$ and K is a cone, we have $\alpha^{-1} y \in K$. Therefore, since $\|y\| = \rho$, we derive from (8.2), (4.3), and (8.3) that

$$\begin{aligned}
\langle x \,|\, p\rangle - \langle x \,|\, y\rangle &= \langle x \,|\, p - y\rangle && (8.5a)\\
&= \langle x - P_K x \,|\, p - y\rangle + \langle P_K x \,|\, p - y\rangle && (8.5b)\\
&= \langle x - P_K x \,|\, \alpha P_K x - y\rangle + \langle \alpha^{-1} p \,|\, p - y\rangle && (8.5c)\\
&= \alpha \underbrace{\langle x - P_K x \,|\, P_K x - \alpha^{-1} y\rangle}_{\geqslant 0 \text{ by } (4.3)} + \alpha^{-1}\langle p \,|\, p - y\rangle && (8.5d)\\
&\geqslant (2\alpha)^{-1}(\|p\|^2 + \|p - y\|^2 - \|y\|^2) && (8.5e)\\
&= (2\alpha)^{-1}(\rho^2 + \|p - y\|^2 - \rho^2) && (8.5f)\\
&= (2\alpha)^{-1}\|p - y\|^2. && (8.5g)
\end{aligned}$$

To summarize, we have shown that $(\forall y \in C)$ $y \neq p \Rightarrow \langle x \,|\, y\rangle < \langle x \,|\, p\rangle$. Combining this, (8.4), and (8.1), we infer from Lemma 4.4(i) that $P_C x = \{p\}$. This and Lemma 4.3(ii) yield the latter assertion, and the proof is complete. ∎

Let us provide some examples.

[9] Due to Lemma 4.3(i), we have $P_K x \neq 0$.

Corollary 8.2 (Projections onto circles) *Let V be a nonzero closed linear subspace of $\mathcal{H}$, let $\rho \in \mathbb{R}_{++}$, and set $C := V \cap \mathrm{S}(0;\rho)$. Then*

$$(\forall x \in \mathcal{H}) \quad P_C x = \begin{cases} C, & \text{if } x \in V^{\perp}; \\ \left\{ \dfrac{\rho}{\|P_V x\|} P_V x \right\}, & \text{otherwise.} \end{cases} \tag{8.6}$$

Proof. Combine Theorem 8.1 and the fact that $V^{\ominus} = V^{\perp}$. ∎

Remark 8.3 Letting $V = \mathcal{H}$ in Corollary 8.2, we see that $C = \mathrm{S}(0;\rho)$, that $V^{\perp} = \{0\}$, that $P_V = \mathrm{Id}$, and that (8.6) becomes

$$(\forall x \in \mathcal{H}) \quad P_C x = \begin{cases} C, & \text{if } x = 0; \\ \left\{ \dfrac{\rho}{\|x\|} x \right\}, & \text{otherwise.} \end{cases} \tag{8.7}$$

Hence, we recover the well-known formula for projectors onto spheres.

Example 8.4 Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_{++}$, and set

$$\mathbf{S}_{\alpha,\beta} := \mathrm{S}(0;\beta) \times \{\alpha\}. \tag{8.8}$$

Then

$$(\forall \mathbf{x} = (x,\xi) \in \boldsymbol{\mathcal{H}}) \quad P_{\mathbf{S}_{\alpha,\beta}} \mathbf{x} = \begin{cases} \mathbf{S}_{\alpha,\beta}, & \text{if } x = 0; \\ \left\{ \left(\dfrac{\beta}{\|x\|} x, \alpha \right) \right\}, & \text{otherwise.} \end{cases} \tag{8.9}$$

Proof. Set $\mathbf{V} := \mathcal{H} \times \{0\}$, which is a nonzero closed linear subspace of $\boldsymbol{\mathcal{H}}$ by (1.1). Let us first observe that

$$\mathbf{V} = \{\mathbf{x} = (x,\xi) \in \boldsymbol{\mathcal{H}} \mid \langle \mathbf{x} \,|\, (0,1) \rangle = 0\} = \{(0,1)\}^{\perp}, \tag{8.10}$$

and thus,

$$(\forall \mathbf{x} = (x,\xi) \in \boldsymbol{\mathcal{H}}) \quad \mathbf{x} \in \mathbf{V}^{\perp} \Leftrightarrow \mathbf{x} \in \mathbb{R}(0,1) \Leftrightarrow x = 0. \tag{8.11}$$

Moreover, it is straightforward to verify that

$$\mathbf{S}_{0,\beta} = \mathbf{V} \cap \mathbf{S}(0;\beta)\,. \tag{8.12}$$

Now fix $\mathbf{x} = (x,\xi) \in \boldsymbol{\mathcal{H}}$. Then, appealing to [3, Example 3.23] and (8.10), we see that $P_{\mathbf{V}}\mathbf{x} = (x,0)$, Combining this, (8.12), and (8.11), we deduce from Corollary 8.2 that

$$P_{\mathbf{S}_{0,\beta}} \mathbf{x} = P_{\mathbf{S}_{0,\beta}} (x,\xi) = \begin{cases} \mathbf{S}_{0,\beta}, & \text{if } x = 0; \\ \left\{ \dfrac{\beta}{\|P_{\mathbf{V}}\mathbf{x}\|} P_{\mathbf{V}}\mathbf{x} \right\}, & \text{otherwise} \end{cases} \tag{8.13a}$$

$$= \begin{cases} \mathbf{S}_{0,\beta}, & \text{if } x = 0; \\ \left\{ \left(\dfrac{\beta}{\|x\|} x, 0 \right) \right\}, & \text{otherwise.} \end{cases} \tag{8.13b}$$

Consequently, since[10] $\mathbf{S}_{\alpha,\beta} = (0,\alpha) + \mathbf{S}_{0,\beta}$, we derive from (8.13b) (applied to the point $(x, \xi - \alpha)$) that

$$P_{\mathbf{S}_{\alpha,\beta}} \mathbf{x} = (0,\alpha) + P_{\mathbf{S}_{0,\beta}} (\mathbf{x} - (0,\alpha)) \tag{8.14a}$$

[10] As the reader can easily verify.

$$= (0,\alpha) + P_{\mathbf{S}_{0,\beta}}(x, \xi - \alpha) \tag{8.14b}$$

$$= \begin{cases} (0,\alpha) + \mathbf{S}_{0,\beta}, & \text{if } x = 0; \\ (0,\alpha) + \left\{ \left(\frac{\beta}{\|x\|} x, 0 \right) \right\}, & \text{otherwise} \end{cases} \tag{8.14c}$$

$$= \begin{cases} \mathbf{S}_{\alpha,\beta}, & \text{if } x = 0; \\ \left\{ \left(\frac{\beta}{\|x\|} x, \alpha \right) \right\}, & \text{otherwise,} \end{cases} \tag{8.14d}$$

as announced in (8.9). ■

Next, we turn to the more complicated case when the point to be projected belongs to the polar cone.

Theorem 8.5 *Let K be a convex cone in $\mathcal{H}$ such that $K \smallsetminus \{0\} \neq \varnothing$, let ρ be in $\mathbb{R}_{++}$, and let $x \in K^{\ominus}$. Suppose that there exists a nonempty subset C of K such that*

$$(\forall y \in C) \quad \|y\| = \rho \tag{8.15}$$

and that

$$K = \operatorname{pos} C. \tag{8.16}$$

Set

$$D := K \cap \mathrm{S}(0;\rho) \quad \textit{and} \quad \kappa := \sup\langle x \,|\, C\rangle. \tag{8.17}$$

Then the following hold:

(i) *Suppose that $P_C x = \varnothing$. Then $P_D x = \varnothing$.*

(ii) *Suppose that $P_C x \neq \varnothing$, and set $E := \mathrm{S}(0;\rho) \cap \operatorname{cone}(\operatorname{conv} P_C x)$. Then the following hold:*

  (a) *$P_C x \subseteq P_D x \subseteq E$ and $\max\langle x \,|\, D\rangle = \max\langle x \,|\, C\rangle$.*

  (b) *Suppose that $\kappa < 0$. Then $P_D x = P_C x$.*

  (c) *Suppose that $\kappa = 0$. Then $P_D x = E$.*

(iii) *$P_C x \neq \varnothing \Leftrightarrow P_D x \neq \varnothing$.*

Proof. We start with a few observations. First, since $K \neq \{0\}$ by assumption, it follows from Remark 3.2 that $D \neq \varnothing$. Next, in view of (8.15) and the assumption that $C \subseteq K$, we have

$$C \subseteq D. \tag{8.18}$$

In turn, because $x \in K^{\ominus}$, we get from (8.16) and Lemma 3.5(i) that

$$\kappa \leqslant 0. \tag{8.19}$$

Finally, by the very definition of D, we see that

$$\text{the vectors in } D \text{ are of equal norm.} \tag{8.20}$$

(i): We prove the contrapositive and therefore assume that there exists

$$u \in P_D x. \tag{8.21}$$

Then, by (8.18), (8.20), (8.21), and Lemma 4.4(i), we obtain

$$\kappa = \sup\langle x \mid C\rangle \leqslant \sup\langle x \mid D\rangle = \langle x \mid u\rangle. \tag{8.22}$$

In turn, combining (8.15), (8.19), (8.22), and the fact that $u \in D = (\operatorname{pos} C) \cap \mathrm{S}(0;\rho)$, we infer from Lemma 4.6(ii)(a) that $P_C x \neq \varnothing$.

(ii)(a): Let us first prove that $P_C x \subseteq P_D x$ and that $\max\langle x \mid D\rangle = \max\langle x \mid C\rangle$. To this end, take $u \in P_C x$ and $y \in D$. Then, because $y \in D \subseteq \operatorname{pos} C$, there exist finite sets $\{\alpha_i\}_{i\in I} \subseteq \mathbb{R}_+$ and $\{x_i\}_{i\in I} \subseteq C$ such that $y = \sum_{i\in I} \alpha_i x_i$. In turn, on the one hand, since $\|y\| = \rho$, we infer from (8.15) and Lemma 4.6(i) that $\sum_{i\in I} \alpha_i \geqslant 1$. On the other hand, since $u \in P_C x$, it follows from (8.15) and Lemma 4.4(i) that

$$\langle x \mid u\rangle = \max\langle x \mid C\rangle = \kappa. \tag{8.23}$$

So altogether, since $(\forall i \in I)\ x_i \in C$, using (8.19), we see that

$$\langle x \mid y\rangle = \sum_{i\in I} \alpha_i \langle x \mid x_i\rangle \leqslant \sum_{i\in I} \alpha_i \kappa = \kappa \sum_{i\in I} \alpha_i \leqslant \kappa = \langle x \mid u\rangle. \tag{8.24}$$

Therefore, since $u \in C \subseteq D$ by (8.18), we derive from (8.24) and (8.23) that

$$\max\langle x \mid D\rangle = \langle x \mid u\rangle = \max\langle x \mid C\rangle. \tag{8.25}$$

Also, appealing to (8.25) and (8.20), we get from Lemma 4.4(i) that $u \in P_D x$, as desired. It now remains to establish the inclusion $P_D x \subseteq E$. To do so, fix $v \in P_D x$. Then, in view of (8.20), Lemma 4.4(i) and (8.25)&(8.23)&(8.19) assert that

$$\langle x \mid v\rangle = \max\langle x \mid D\rangle = \max\langle x \mid C\rangle \leqslant 0. \tag{8.26}$$

Thus, since

$$v \in D = (\operatorname{pos} C) \cap \mathrm{S}(0;\rho), \tag{8.27}$$

it follows from (8.15) and Lemma 4.6(ii)(b) that $v \in \mathrm{S}(0;\rho) \cap \operatorname{cone}(\operatorname{conv} P_C x) = E$, as claimed.

(ii)(b): Consider the element $v \in P_D x$ of the proof of (ii)(a). Combining (8.15)&(8.26)&(8.27) and the assumption that $\kappa < 0$, we derive from Lemma 4.6(ii)(c) that $v \in P_C x$, and hence, $P_D x \subseteq P_C x$. Consequently, since $P_C x \subseteq P_D x$ by (ii)(a), the assertion follows.

(ii)(c): According to (ii)(a), it suffices to show that $E \subseteq P_D x$. Towards this end, take $w \in E$ and $y \in D$. By the very definition of E, there exist finite sets $\{\beta_j\}_{j\in J} \subseteq \mathbb{R}_{++}$ and $\{x_j\}_{j\in J} \subseteq P_C x$ such that $w = \sum_{j\in J} \beta_j x_j$. In turn, since $\{x_j\}_{j\in J} \subseteq P_C x$, we get from (8.15) and Lemma 4.4(i) that $(\forall j \in J)\ \langle x \mid x_j\rangle = \kappa = 0$, from which and (8.25) it follows that

$$\langle x \mid w\rangle = \sum_{j\in J} \beta_j \langle x \mid x_j\rangle = 0 = \kappa = \max\langle x \mid D\rangle. \tag{8.28}$$

Consequently, since $w \in E \subseteq D$ by the very definitions of E and D, invoking (8.20) and Lemma 4.4(i) once more, we conclude that $w \in P_D x$, as required.

(iii): Combine (i) and (ii)(a). ■

We are now ready for the main result of this section which provides a formula for the projector of a finitely generated cone and a sphere.

Corollary 8.6 (cone intersected with sphere) *Let $\{x_i\}_{i\in I}$ be a nonempty finite subset of $\mathcal{H}$, let $\rho \in \mathbb{R}_{++}$, and let $x \in \mathcal{H}$. Set*

$$K := \sum_{i\in I} \mathbb{R}_+ x_i, \quad C := K \cap \mathrm{S}(0;\rho), \quad \kappa := \max_{i\in I}\langle x \mid x_i\rangle, \quad \text{and } I(x) := \{i \in I \mid \langle x \mid x_i\rangle = \kappa\}. \tag{8.29}$$

Suppose that $(\forall i \in I)\ \|x_i\| = \rho$. *Then*

$$P_C x = \begin{cases} \left\{ \dfrac{\rho}{\|P_K x\|} P_K x \right\}, & \textit{if } \kappa > 0; \\ \mathrm{S}(0;\rho) \cap \operatorname{cone}\Big(\operatorname{conv}\{x_i\}_{i \in I(x)}\Big), & \textit{if } \kappa = 0; \\ \{x_i\}_{i \in I(x)}, & \textit{if } \kappa < 0. \end{cases} \tag{8.30}$$

Proof. Set $X := \{x_i\}_{i\in I}$. First, it follows from Example 3.11 that K is a nonempty closed convex cone. In addition, Lemma 3.5(i) (applied to $\{x_i\}_{i\in I}$) implies that

$$x \in K^{\ominus} \Leftrightarrow \kappa = \max_{i \in I} \langle x \mid x_i \rangle \leqslant 0. \tag{8.31}$$

Next, due to our assumption, Lemma 4.4(i) yields

$$P_X x = \{x_i\}_{i \in I(x)} \neq \varnothing. \tag{8.32}$$

Let us now identify $P_C x$ in each of the following conceivable cases:

(A) $\kappa > 0$: Then, by (8.31), we have $x \in \mathcal{H} \smallsetminus K^{\ominus}$, and hence, Theorem 8.1(ii) asserts that $P_C x = \{(\rho/\|P_K x\|) P_K x\}$.

(B) $\kappa = 0$: Using Theorem 8.5(ii)(c) (with the set C being $X = \{x_i\}_{i\in I}$) and (8.32), we obtain $P_C x = \mathrm{S}(0;\rho) \cap \operatorname{cone}(\operatorname{conv}\{x_i\}_{i\in I(x)})$.

(C) $\kappa < 0$: Invoking Theorem 8.5(ii)(b) and (8.32), we immediately have $P_C x = \{x_i\}_{i\in I(x)}$. ■

Remark 8.7 Consider the setting of Corollary 8.6. Since $\{x_i\}_{i\in I(x)} \subseteq \mathrm{S}(0;\rho) \cap \operatorname{cone}(\operatorname{conv}\{x_i\}_{i\in I(x)})$ by the assumption that $\|x_i\| \equiv \rho$, we see that

$$\mathrm{s} \colon \mathcal{H} \to \mathcal{H} : x \mapsto \begin{cases} \dfrac{\rho}{\|P_K x\|} P_K x, & \text{if } \max\limits_{i \in I} \langle x \mid x_i \rangle > 0; \\ \mathrm{s}(x) \in \{x_i\}_{i \in I(x)}, & \text{otherwise} \end{cases} \tag{8.33}$$

is a selection of P_C.

Example 8.8 Consider the setting of Theorem 6.1. Set

$$C := K \cap \mathrm{S}(0;1)\,,\ \kappa := \max_{i \in I} \langle x \mid e_i \rangle,\ I(x) := \{i \in I \mid \langle x \mid e_i \rangle = \kappa\},\ \text{and } \lambda := \sqrt{\sum_{i \in I} (\max\{\langle x \mid e_i \rangle, 0\})^2}. \tag{8.34}$$

Then

$$P_C x = \begin{cases} \left\{ \lambda^{-1} \sum\limits_{i \in I} \max\{\langle x \mid e_i \rangle, 0\}\, e_i \right\}, & \text{if } \kappa > 0; \\ \left\{ \sum\limits_{i \in I(x)} \alpha_i e_i \;\middle|\; \{\alpha_i\}_{i \in I(x)} \subseteq \mathbb{R}_+ \text{ such that } \sum\limits_{i \in I(x)} \alpha_i^2 = 1 \right\}, & \text{if } \kappa = 0; \\ \{e_i\}_{i \in I(x)}, & \text{if } \kappa < 0. \end{cases} \tag{8.35}$$

Proof. Since $P_K x = \sum_{i\in I} \max\{\langle x \mid e_i\rangle, 0\}\, e_i$ by (6.2), we obtain

$$\|P_K x\|^2 = \left\| \sum_{i \in I} \max\{\langle x \mid e_i \rangle, 0\}\, e_i \right\|^2 = \sum_{i \in I} (\max\{\langle x \mid e_i \rangle, 0\})^2 = \lambda^2. \tag{8.36}$$

Next, let us show that

$$\mathrm{S}(0;1)\cap\mathrm{cone}\Big(\mathrm{conv}\{e_i\}_{i\in I(x)}\Big)=\left\{\sum_{i\in I(x)}\alpha_i e_i\;\middle|\;\{\alpha_i\}_{i\in I(x)}\subseteq\mathbb{R}_+\text{ such that }\sum_{i\in I(x)}\alpha_i^2=1\right\}. \tag{8.37}$$

To this end, denote the set on the right-hand side of (8.37) by D. Take $y\in\mathrm{S}(0;1)\cap\mathrm{cone}(\mathrm{conv}\{e_i\}_{i\in I(x)})$. Then there exist $\lambda\in\mathbb{R}_{++}$ and $\{\alpha_i\}_{i\in I(x)}\subseteq\mathbb{R}_+$ such that $y=\lambda\sum_{i\in I(x)}\alpha_i e_i=\sum_{i\in I(x)}(\lambda\alpha_i)e_i$. Furthermore, since $\{e_i\}_{i\in I(x)}$ is an orthonormal set, we get $1=\|y\|^2=\|\sum_{i\in I(x)}(\lambda\alpha_i)e_i\|^2=\sum_{i\in I(x)}(\lambda\alpha_i)^2$. Hence $y\in D$. Conversely, fix $z\in D$, say $z=\sum_{i\in I(x)}\beta_i e_i$, where $\{\beta_i\}_{i\in I(x)}\subseteq\mathbb{R}_+$ satisfying $\sum_{i\in I(x)}\beta_i^2=1$, and set $\beta:=\sum_{i\in I(x)}\beta_i$. It is clear that $\beta>0$, and therefore, $z=\beta\sum_{i\in I(x)}(\beta_i/\beta)e_i\in\mathrm{cone}(\mathrm{conv}\{e_i\}_{i\in I(x)})$. In turn, because $\|z\|^2=\sum_{i\in I(x)}\beta_i^2=1$, it follows that $z\in\mathrm{S}(0;1)\cap\mathrm{cone}(\mathrm{conv}\{e_i\}_{i\in I(x)})$. Thus (8.37) holds. Consequently, using (6.2)&(8.36)&(8.37), we obtain (8.35) via Corollary 8.6. ■

The following nice result was mentioned in [12, Example 5.5.2 and Problem 5.6.14].

Example 8.9 (Lange) Suppose that $\mathcal{H}=\mathbb{R}^N$, that $I=\{1,\ldots,N\}$, and that $(e_i)_{i\in I}$ is the canonical orthonormal basis of $\mathcal{H}$. Set

$$K:=\mathbb{R}^N_+\quad\text{and}\quad C:=K\cap\mathrm{S}(0;1)\,. \tag{8.38}$$

Now let $x=(\xi_i)_{i\in I}\in\mathcal{H}$; set $\kappa:=\max_{i\in I}\xi_i$, $I(x):=\{i\in I\mid\xi_i=\kappa\}$, and $x_+:=(\max\{\xi_i,0\})_{i\in I}$. Then

$$P_C x=\begin{cases}\left\{\dfrac{1}{\|x_+\|}x_+\right\}, & \text{if }\kappa>0;\\[2mm] \left\{\displaystyle\sum_{i\in I(x)}\alpha_i e_i\;\middle|\;\{\alpha_i\}_{i\in I(x)}\subseteq\mathbb{R}_+\text{ such that }\sum_{i\in I(x)}\alpha_i^2=1\right\}, & \text{if }\kappa=0;\\[2mm] \{e_i\}_{i\in I(x)}, & \text{if }\kappa<0.\end{cases} \tag{8.39}$$

Proof. Because $(\forall i\in I)$ $\langle x\mid e_i\rangle=\xi_i$ and $\|x_+\|^2=\sum_{i\in I}(\max\{\xi_i,0\})^2$, (8.39) therefore follows from Example 8.8. ■

9 Further examples

In this section, we provide further examples based on the Lorentz cone and on the cone of positive semidefinite matrices.

Example 9.1 Let α and ρ be in $\mathbb{R}_{++}$, let

$$\boldsymbol{K}_\alpha=\{(x,\xi)\in\mathcal{H}\oplus\mathbb{R}\mid\|x\|\leqslant\alpha\xi\} \tag{9.1}$$

be the Lorentz cone of parameter α of Example 3.6, set $\boldsymbol{C}:=\boldsymbol{K}_\alpha\cap\mathbf{S}(0;\rho)$, and let $\boldsymbol{x}=(x,\xi)\in\boldsymbol{\mathcal{H}}$. Then

$$P_{\boldsymbol{C}}\boldsymbol{x}=\begin{cases}\left\{\dfrac{\rho}{\|\boldsymbol{x}\|}\boldsymbol{x}\right\}, & \text{if }\|x\|\leqslant\alpha\xi\text{ and }\xi>0;\\[2mm] \left\{\dfrac{\rho}{\sqrt{1+\alpha^2}}\left(\dfrac{\alpha x}{\|x\|},1\right)\right\}, & \text{if }\|x\|>\max\{\alpha\xi,-\xi/\alpha\}\text{ or }[\,x\neq0\text{ and }\|x\|\leqslant-\xi/\alpha\,];\\[2mm] \mathrm{S}(0;\beta)\times\{\beta/\alpha\}, & \text{if }x=0\text{ and }\xi<0;\\ \boldsymbol{C}, & \text{if }(x,\xi)=(0,0).\end{cases} \tag{9.2}$$

Proof. Set

$$\beta := \frac{\rho\alpha}{(1+\alpha^2)^{1/2}} \in \mathbb{R}_{++}, \tag{9.3}$$

$\boldsymbol{C}_{\alpha,\beta} := \mathrm{S}(0;\beta) \times \{\beta/\alpha\}$, and $\kappa := \max\langle \boldsymbol{x} \mid \boldsymbol{C}_{\alpha,\beta}\rangle$. Then it is readily verified that

$$(\forall \boldsymbol{y} \in \boldsymbol{C}_{\alpha,\beta}) \quad \|\boldsymbol{y}\| = \rho, \tag{9.4}$$

and due to Lemma 2.3,

$$\kappa = \beta\|x\| + \xi\beta/\alpha. \tag{9.5}$$

Furthermore, by Example 3.6,

$$\boldsymbol{K}_\alpha = \operatorname{pos} \boldsymbol{C}_{\alpha,\beta} = \operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta}) \cup \{\boldsymbol{0}\}, \tag{9.6}$$

and by Example 8.4 (applied to $\boldsymbol{C}_{\alpha,\beta}$), we have

$$\varnothing \neq P_{\boldsymbol{C}_{\alpha,\beta}}\boldsymbol{x} = \begin{cases} \boldsymbol{C}_{\alpha,\beta}, & \text{if } x = 0; \\ \left\{\left(\dfrac{\beta}{\|x\|}x, \dfrac{\beta}{\alpha}\right)\right\}, & \text{otherwise.} \end{cases} \tag{9.7}$$

Let us now identify $P_{\boldsymbol{C}}\boldsymbol{x}$ in the following conceivable cases:

(A) $\|x\| > -\xi/\alpha$: Then $\kappa > 0$ by (9.5), and so by (9.6) and Lemma 3.5(i), $\boldsymbol{x} \in \boldsymbol{\mathcal{H}} \smallsetminus \boldsymbol{K}_\alpha^{\ominus}$. In turn, it follows from Theorem 8.1(ii) (applied to $\boldsymbol{C} = \boldsymbol{K}_\alpha \cap \mathbf{S}(0;\rho)$) that

$$P_{\boldsymbol{C}}\boldsymbol{x} = \left\{\frac{\rho}{\|P_{\boldsymbol{K}_\alpha}\boldsymbol{x}\|} P_{\boldsymbol{K}_\alpha}\boldsymbol{x}\right\}. \tag{9.8}$$

To evaluate $P_{\boldsymbol{C}}\boldsymbol{x}$ further, we consider two subcases:

(A.1) $\|x\| \leqslant \alpha\xi$: Then $\boldsymbol{x} \in \boldsymbol{K}_\alpha$ by (9.1), and so $P_{\boldsymbol{K}_\alpha}\boldsymbol{x} = \boldsymbol{x}$, which yields $P_{\boldsymbol{C}}\boldsymbol{x} = \{(\rho/\|\boldsymbol{x}\|)\boldsymbol{x}\}$.

(A.2) $\|x\| > \alpha\xi$: Then, according to [3, Exercise 29.11],

$$P_{\boldsymbol{K}_\alpha}\boldsymbol{x} = P_{\boldsymbol{K}_\alpha}(x,\xi) = \frac{\alpha\|x\| + \xi}{1+\alpha^2}\left(\frac{\alpha x}{\|x\|}, 1\right), \tag{9.9}$$

and since $\alpha\|x\| + \xi > 0$, it follows that

$$\|P_{\boldsymbol{K}_\alpha}\boldsymbol{x}\| = \frac{\alpha\|x\| + \xi}{1+\alpha^2}\left\|\left(\frac{\alpha x}{\|x\|}, 1\right)\right\| = \frac{\alpha\|x\| + \xi}{1+\alpha^2}\sqrt{\left\|\frac{\alpha x}{\|x\|}\right\|^2 + 1} = \frac{\alpha\|x\| + \xi}{\sqrt{1+\alpha^2}}. \tag{9.10}$$

Hence, combining (9.8)&(9.9)&(9.10), we get

$$P_{\boldsymbol{C}}\boldsymbol{x} = \left\{\frac{\rho}{\sqrt{1+\alpha^2}}\left(\frac{\alpha x}{\|x\|}, 1\right)\right\}. \tag{9.11}$$

(B) $\|x\| = -\xi/\alpha$: Then $\kappa = 0$ by (9.5), and invoking (9.4)&(9.6)&(9.7), Theorem 8.5(ii)(c) asserts that

$$P_{\boldsymbol{C}}\boldsymbol{x} = \mathbf{S}(0;\rho) \cap \operatorname{cone}(\operatorname{conv} P_{\boldsymbol{C}_{\alpha,\beta}}\boldsymbol{x}). \tag{9.12}$$

We consider two subcases:

(B.1) $x = 0$: Then $\xi = 0$ and so $\boldsymbol{x} = (x,\xi) = 0$. Moreover, due to (9.7), $P_{\boldsymbol{C}_{\alpha,\beta}}\boldsymbol{x} = \boldsymbol{C}_{\alpha,\beta}$. Therefore, by (9.6) and (9.12),

$$\boldsymbol{C} = \boldsymbol{K}_\alpha \cap \mathbf{S}(0;\rho) \tag{9.13a}$$

$$= (\operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta}) \cup \{\boldsymbol{0}\}) \cap \mathbf{S}(0;\rho) \tag{9.13b}$$

$$= \operatorname{cone}(\operatorname{conv} \boldsymbol{C}_{\alpha,\beta}) \cap \mathbf{S}(0;\rho) \tag{9.13c}$$
$$= \operatorname{cone}(\operatorname{conv} P_{\boldsymbol{C}_{\alpha,\beta}}\boldsymbol{x}) \cap \mathbf{S}(0;\rho) \tag{9.13d}$$
$$= P_{\boldsymbol{C}}\boldsymbol{x}. \tag{9.13e}$$

(B.2) $x \neq 0$: Then (9.7) yields $P_{\boldsymbol{C}_{\alpha,\beta}}\boldsymbol{x} = \{(\beta x/\|x\|, \beta/\alpha)\}$. In turn, since $\|(\beta x/\|x\|, \beta/\alpha)\| = \rho$ by (9.3) and a simple computation, we obtain from (9.12) and Fact 3.3(i) that

$$P_{\boldsymbol{C}}\boldsymbol{x} = \mathbf{S}(0;\rho) \cap \operatorname{cone}(\operatorname{conv} P_{\boldsymbol{C}_{\alpha,\beta}}\boldsymbol{x}) \tag{9.14a}$$
$$= \mathbf{S}(0;\rho) \cap \left(\mathbb{R}_{++}\left(\frac{\beta x}{\|x\|}, \frac{\beta}{\alpha}\right)\right) \tag{9.14b}$$
$$= \left\{\left(\frac{\beta x}{\|x\|}, \frac{\beta}{\alpha}\right)\right\} \tag{9.14c}$$
$$= \left\{\frac{\beta}{\alpha}\left(\frac{\alpha x}{\|x\|}, 1\right)\right\} \tag{9.14d}$$
$$= \left\{\frac{\rho}{\sqrt{1+\alpha^2}}\left(\frac{\alpha x}{\|x\|}, 1\right)\right\}. \tag{9.14e}$$

(C) $\|x\| < -\xi/\alpha$: Then $\kappa < 0$ by (9.5), and so, in view of (9.4)&(9.6)&(9.7), we deduce from Theorem 8.5(ii)(b) that $P_{\boldsymbol{C}}\boldsymbol{x} = P_{\boldsymbol{C}_{\alpha,\beta}}\boldsymbol{x}$. Hence, by (9.7) and (9.3), we get

$$P_{\boldsymbol{C}}\boldsymbol{x} = \begin{cases} \boldsymbol{C}_{\alpha,\beta}, & \text{if } x = 0; \\ \left\{\left(\frac{\beta}{\|x\|}x, \frac{\beta}{\alpha}\right)\right\}, & \text{if } x \neq 0 \end{cases} \tag{9.15a}$$
$$= \begin{cases} \boldsymbol{C}_{\alpha,\beta}, & \text{if } x = 0; \\ \left\{\frac{\rho}{\sqrt{1+\alpha^2}}\left(\frac{\alpha x}{\|x\|}, 1\right)\right\}, & \text{if } x \neq 0. \end{cases} \tag{9.15b}$$

To sum up, we have shown that

$$P_{\boldsymbol{C}}\boldsymbol{x} = \begin{cases} \left\{\frac{\rho}{\|\boldsymbol{x}\|}\boldsymbol{x}\right\}, & \text{if } -\xi/\alpha < \|x\| \leqslant \alpha\xi; \\ \left\{\frac{\rho}{\sqrt{1+\alpha^2}}\left(\frac{\alpha x}{\|x\|}, 1\right)\right\}, & \text{if } \|x\| > \max\{\alpha\xi, -\xi/\alpha\} \text{ or } [\, x \neq 0 \text{ and } \|x\| \leqslant -\xi/\alpha\,]; \\ \boldsymbol{C}_{\alpha,\beta}, & \text{if } x = 0 \text{ and } 0 < -\xi; \\ \boldsymbol{C}, & \text{if } (x,\xi) = (0,0) \end{cases} \tag{9.16a}$$
$$= \begin{cases} \left\{\frac{\rho}{\|\boldsymbol{x}\|}\boldsymbol{x}\right\}, & \text{if } \|x\| \leqslant \alpha\xi \text{ and } \xi > 0; \\ \left\{\frac{\rho}{\sqrt{1+\alpha^2}}\left(\frac{\alpha x}{\|x\|}, 1\right)\right\}, & \text{if } \|x\| > \max\{\alpha\xi, -\xi/\alpha\} \text{ or } [\, x \neq 0 \text{ and } \|x\| \leqslant -\xi/\alpha\,]; \\ \mathrm{S}(0;\beta) \times \{\beta/\alpha\}, & \text{if } x = 0 \text{ and } \xi < 0; \\ \boldsymbol{C}, & \text{if } (x,\xi) = (0,0), \end{cases} \tag{9.16b}$$

as announced in (9.2). ■

Example 9.2 Suppose that $\mathcal{H} = \mathbb{S}^N$ is the Hilbert space of symmetric matrices of Section 5. Set $K := \mathbb{S}^N_+$, let $\rho \in \mathbb{R}_{++}$, and set $\mathsf{C} := K \cap \mathsf{S}(0;\rho)$. In addition, let $A \in \mathcal{H}$, and let $U \in \mathbb{U}^N$ be such that $A = U(\operatorname{Diag}\lambda(A))U^\mathsf{T}$; set

$$\mathsf{D} := \{V(\operatorname{Diag}(\rho,0,\dots,0))V^\mathsf{T} \mid V \in \mathbb{U}^N \text{ such that } A = V(\operatorname{Diag}\lambda(A))V^\mathsf{T}\} \tag{9.17}$$

and

$$\mathsf{E} := \mathsf{S}(0;\rho) \cap \operatorname{cone}(\operatorname{conv}\mathsf{D}). \tag{9.18}$$

Then

$$P_\mathsf{C}A = \begin{cases} \left\{\dfrac{\rho}{\|(\lambda(A))_+\|}U(\operatorname{Diag}(\lambda(A))_+)U^\mathsf{T}\right\}, & \text{if } \lambda_1(A) > 0; \\ \mathsf{E}, & \text{if } \lambda_1(A) = 0; \\ \mathsf{D}, & \text{if } \lambda_1(A) < 0. \end{cases} \tag{9.19}$$

Proof. Set

$$\mathsf{C}_\rho := \left\{B \in \mathbb{S}^N_+ \;\middle|\; \operatorname{rank} B = 1 \text{ and } \|B\|_\mathsf{F} = \rho\right\}. \tag{9.20}$$

It then follows from Lemma 5.4(iii) that

$$\max\langle A \mid \mathsf{C}_\rho\rangle = \rho\lambda_1(A) \quad \text{and} \quad P_{\mathsf{C}_\rho}A = \mathsf{D}. \tag{9.21}$$

Let us now consider all conceivable cases:

(A) $\lambda_1(A) > 0$: Then $\max\langle A \mid \mathsf{C}_\rho\rangle > 0$, and thus, by Lemma 5.4(i) and Lemma 3.5(i), we obtain $A \in \mathcal{H} \smallsetminus K^\ominus$. Therefore, since $\{0\} \neq K$ is a nonempty closed convex cone, we infer from Theorem 8.1(ii) and Lemma 5.2 that

$$P_\mathsf{C}A = \left\{\frac{\rho}{\|P_KA\|_\mathsf{F}}P_KA\right\} = \left\{\frac{\rho}{\|(\lambda(A))_+\|}U(\operatorname{Diag}(\lambda(A))_+)U^\mathsf{T}\right\}. \tag{9.22}$$

(B) $\lambda_1(A) \leqslant 0$: Then $\max\langle A \mid \mathsf{C}_\rho\rangle \leqslant 0$. Since $(\forall B \in \mathsf{C}_\rho)\ \|B\|_\mathsf{F} = \rho$ and, by Lemma 5.4(i), $K = \operatorname{pos}\mathsf{C}_\rho$, it follows from Theorem 8.5(ii)(b)&(ii)(c) and (9.21) that

$$P_\mathsf{C}A = \begin{cases} P_{\mathsf{C}_\rho}A, & \text{if } \max\langle A \mid \mathsf{C}_\rho\rangle < 0; \\ \mathsf{S}(0;\rho) \cap \operatorname{cone}(\operatorname{conv} P_{\mathsf{C}_\rho}A), & \text{if } \max\langle A \mid \mathsf{C}_\rho\rangle = 0 \end{cases} \tag{9.23a}$$

$$= \begin{cases} \mathsf{D}, & \text{if } \lambda_1(A) < 0; \\ \mathsf{E}, & \text{if } \lambda_1(A) = 0, \end{cases} \tag{9.23b}$$

which completes the proof. ■

Remark 9.3 Consider the setting of Example 9.2. Since $U(\operatorname{Diag}(\rho,0,\dots,0))U^\mathsf{T} \in \mathsf{D} \subseteq \mathsf{E}$, we see that

$$\mathsf{s}\colon \mathcal{H} \to \mathcal{H} : A \mapsto \begin{cases} \dfrac{\rho}{\|(\lambda(A))_+\|}U(\operatorname{Diag}(\lambda(A))_+)U^\mathsf{T}, & \text{if } \lambda_1(A) > 0; \\ U(\operatorname{Diag}(\rho,0,\dots,0))U^\mathsf{T}, & \text{otherwise} \end{cases} \tag{9.24}$$

is a selection of P_C.

10 Copositive matrices: a numerical experiment

In this final section, N is a strictly positive integer and M is a symmetric matrix in $\mathbb{R}^{N\times N}$. Recall that M is *copositive* if $(\forall x \in \mathbb{R}^N_+)\ \langle x \mid Mx\rangle \geqslant 0$; or, equivalently,

$$\mu(M) := \min_{x\in\mathbb{R}^N_+\cap \mathrm{S}(0;1)} \tfrac{1}{2}\langle x \mid Mx\rangle \geqslant 0. \tag{10.1}$$

For further information on copositive matrices, we refer the reader to the surveys [6, 11] and references therein. In view of (10.1), testing copositivity of M amounts to

$$\underset{x\in\mathbb{R}^N_+\cap \mathrm{S}(0;1)}{\text{minimize}}\ \tfrac{1}{2}\langle x \mid Mx\rangle. \tag{10.2}$$

Now, set $C := \mathbb{R}^N_+ \cap \mathrm{S}(0;1)$, set $f\colon \mathbb{R}^N \to \mathbb{R} : x \mapsto (1/2)\langle x \mid Mx\rangle$, and set $g := \iota_C$ which is the indicator function of C. Note that neither f nor g is convex; however, ∇f is Lipschitz continuous with the operator norm $\|M\|$ (computed as the largest singular value of M) being a suitable Lipschitz constant. The projection onto C is computed using (8.39). In turn, (10.2) can be written as

$$\underset{x\in\mathbb{R}^N}{\text{minimize}}\ f(x) + g(x). \tag{10.3}$$

To solve this problem, we compared the *Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)* (see [4]), the *Projected Gradient Method (PGM)* (see [2, 5]), the algorithm presented in [12, Example 5.5.2] by Lange, the *Douglas–Rachford Algorithm (DRA)* variant presented in [14] by Li and Pong, and the regular DRA for solving (10.3) when $N \in \{2, 3, 4\}$. For each $N \in \{2, 3, 4\}$, using the copositivity criteria for matrices of order up to four (see, e.g., [8, 16]), we randomly generate 100 copositive matrices (group A) together with 100 non-copositive (group B) ones. For each algorithm, if $(x_n)_{n\in\mathbb{N}}$ is the sequence generated, then we terminate the algorithm when

$$\frac{\|x_n - x_{n-1}\|}{\max\{\|x_{n-1}\|, 1\}} < 10^{-8}. \tag{10.4}$$

The maximum allowable number of iterations is 1000. For each matrix M in group A (respectively, group B), we declare success if $\mu(M) \geqslant 0$ (respectively, $\mu(M) < 0$). We also record the average of the number of iterations until success of each algorithm. The results, obtained using `Matlab`, are reported in Table 1.

Size	Copositive	FISTA		PGM		Lange		Li–Pong		DR	
		succ	avg iter	succ	avg iter	succ	avg iter	succ	avg iter	succ	avg iter
2×2	Yes	100	5	100	5	100	89	100	94	96	23
	No	97	15	99	12	91	92	93	87	53	89
3×3	Yes	100	27	100	24	100	91	100	232	95	63
	No	96	30	98	24	86	93	95	162	31	214
4×4	Yes	100	60	100	62	100	90	100	482	85	126
	No	100	51	100	45	94	95	100	264	11	114

Table 1 Detecting whether a matrix is copositive using a variety of algorithms.

Finally, let us apply the algorithms to the well-known Horn matrix

$$H := \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}, \tag{10.5}$$

which is copositive with $\mu(H) = 0$ (see [9, Equation (3.5)]). For each algorithm, we record the number of iterations and the value of f at the point that the algorithm is terminated. The results are recorded in Table 2.

FISTA		PGM		Lange		Li–Pong		DR	
fval	iter	fval	iter	fval	iter	fval	iter	fval	iter
3.5230e−17	11	2.8297e−20	10	2.9979e−07	95	1.4912e−14	170	0.0584	13

Table 2 Detecting copositivity of the Horn matrix.

We acknowledge that these algorithms might get stuck at points that are not solutions and that the outcome might depend on the starting points; moreover, a detailed complexity analysis is absent. There are thus various research opportunities to improve the current results. Nonetheless, our preliminary results indicate that FISTA and PGM are potentially significant contenders for numerically testing copositivity.

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